

المملكة العربية السعودية
وزارة التعليم العالي
جامعة أم القرى
كلية العلوم التطبيقية
قسم العلوم الرياضية



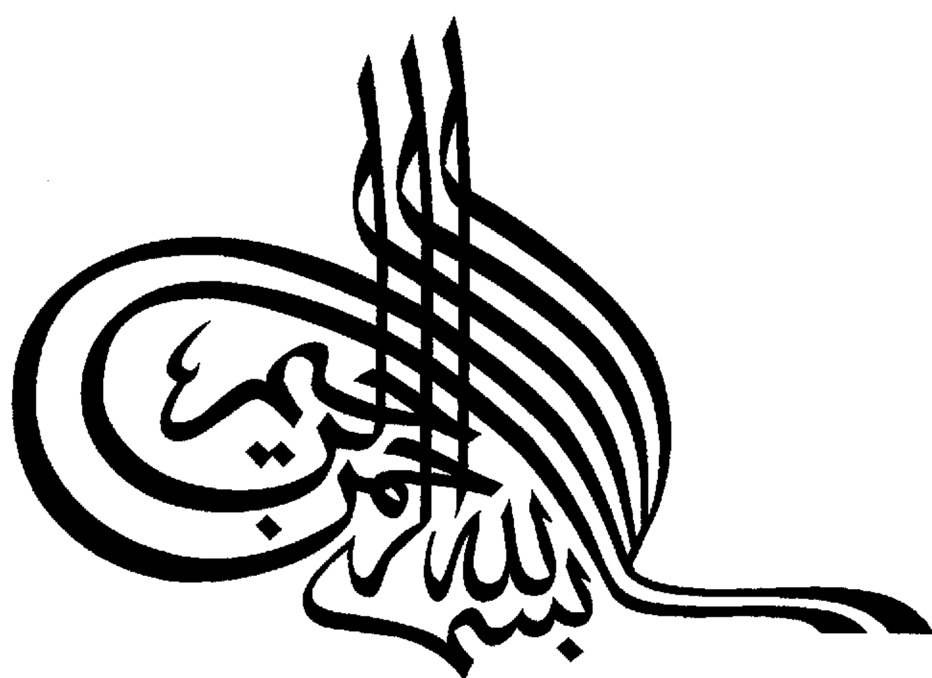
دراسة نظرية وعددية لمعادلة كورتيج دي فريس المعممة في السوليتونات

إعداد الطالبة
شريفة عيسى علي الحازمي

إشراف
الدكتور / كمال محمد عبد الله حميدة

بحث تكميلي لمتطلبات الحصول على درجة الماجستير في الرياضيات التطبيقية

١٤٢٣هـ/٢٠٠٢م



وزارة التعليم العالي
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بسم الله الرحمن الرحيم

محضر مناقشة رسالة (ماجستير)

الحمد لله والصلاة والسلام على رسول الله صلى الله عليه وسلم وعلى آله وصحبه ومن والاه ففي تمام الساعة ٩ من يوم الإثنين الموافق ٢٠/١٠/١٤٢٧هـ اجتمعت اللجنة المشكلة بقرار مجلس كلية رقم ١٩ وفي جلسته ..... بتاريخ ٢٠/٧/١٤٢٧هـ والمكونة من أصحاب السعادة :

م	الاسم	الجهة التابع لها	وضع المناقش في اللجنة
١	د / محمد كامل أحمد خليفة	كلية العلوم - جامعة المنوفية	مناقش خارجي
	د. عبد الفتاح قاري بخاري	رئيس قسم العلوم الرياضية	مناقش داخلي
٣	د. كمال محمد عبد الله صبرة	كلية العلوم - جامعة أم القرى	مشرف على الرسالة

لمناقشة الرسالة المقدمة لنيل درجة الماجستير في العلوم الرياضية ..
من الطالب / الطالبة ...شريفة حسين بخاري...
تخصص .... بكلية العلوم التطبيقية بمكة المكرمة بجامعة أم القرى .
بعنوان ...[illegible]...
...[illegible]...
وبعد الانتهاء اللجنة من المناقشة في الساعة ...١٢... والتداول فيما بينها .
وبناء على موقف الطالب / الطالبة أثناء المناقشة .
أوصت اللجنة بمنح درجة الماجستير في العلوم الرياضية بتقدير ...ممتاز... بالدرجة ...٩٥... الموضحة . والله الموفق .
ما اكتسبه من تقديرات بالسنة المنهجية ...٩٢,٦٦ / ٢٠٠...
ما اكتسبه من الساعات المعتمدة المطلوبة لهذه الدرجة ...٢٤...
ملاحظات
....
....

م	الاسم	الدرجة	التوقيع
١	د / محمد كامل أحمد خليفة	٩٥	[signature]
	د. عبد الفتاح قاري بخاري	٩٥	[signature]
٣	د. كمال محمد عبد الله صبرة	٩٦	[signature]
	مجموع الدرجات	٢٨٦	
	المتوسط	٩٥,٢	

رئيس قسم العلوم الرياضية

د . عبدالفتاح بن قاري بخاري

شرف...

إهداء

إلى القلوب التي لا تكل من العطاء

إلى النجوم الساطعة في كبد السماء

إلى النهر المتدفق بالحب والحياة

والدي . . . والدتي . . . زوجي

ولكل من جعل العلم غاية له ووسيلة

إليكم أهدي هذا الجهد المتواضع

الباحثة

الشكر

سبحانك اللهم المتوحد في الجلال بكمال الجمال تعظيمًا وتكبيرًا، المتفرد بتصريف الأحوال على التفصيل والإجمال تقديرًا وتدبيرًا، المتعالي بعظمتك ومجدك. أحمدك الهي حمدًا كثيرًا خالدًا مع خلودك لامنتهى له دون علمك ولامنتهى له دون مشيئتك ولا أجر لقائله إلا رضاك. أنعمت علينا بنعمة الإسلام وأرسلت إلينا خير الأنام نبينا محمد عليه السلام المتنزل عليه خير البيان، شفاءً للصدور ونورًا للأذهان. فلك الهي يا عظيم الشأن أرفع أكفي شاكرةً على أن سخرت لي طريقًا التمس فيه علمًا هو بالنسبة لعلمك شيئاً يسيرًا فعلمتني مالم أكن أعلم سبحانك فأنت (الرحمن * علم القرآن * خلق الإنسان * علمه البيان) ومننت علي بأفضل العلوم لأنهل منه وأزداد بذلك إيمانًا ويقيناً فلك الحمد على ما أعطيتني وهديتني وللخوض في بحور آياتك سيرتني.

إنه لمن باب الوفاء والتقدير وتوخي الأمانة وقول الصدق أن أعترف بفضل وجميل كل من أسهم في سبيل إخراج هذا البحث إلى حيز النور. فأقدم كل شكري وعرفاني للدكتور /كمال محمد حميدة مشرفي وأستاذي الجليل الذي منحني أغلى أوقاته، وما فتئ يحثني على الكمال – والكمال لله وحده – وييسر لي السبل ويمدني بالمراجع النادرة، فأعمل فكره من أجل إخراج هذه الرسالة بهذه الصورة فله على صاحبة هذا البحث ما يعجز القلم عن وصفه، والفضل

عن ذكره ، جزاه الله خير الجزاء ، وأمد في عمره وعمر أمثاله ليظل المنهل العذب الذي ينهل منه كل طالب علم ومعرفة . ولأستاذي الجليل الدكتور /عبد الفتاح قاري أياد بيضاء في حياتي العلمية منذ دراستي في المرحلة المنهجية فكان نعم الأستاذ المخلص الناصح والموجه لي ، فجزاه الله عني خير الجزاء ، ووفقه لما يحبه ويرضاه .

هذا وأتقدم بالشكر الجزيل لسعادة الدكتور / أحمد خليفة الذي تفضل بالحضور لمناقشة هذه الرسالة وتحمل عبء قراءتها ،ولسعادة الدكتور / أحمد خماش لمساعدته في إتمام إجراءات حضور المشرف . هذا مع كثرة مشاغلهم فجزاهم الله جميعاً أفضل الجزاء .

وأخيراً أتوجه بخالص الشكر الى جامعة أم القرى التي شرفت بطلب العلم فيها حتى وصلت إلى المرحلة التي أنا فيها الآن ، والشكر موصول لكلية العلوم التطبيقية وأعضاء قسم العلوم الرياضية بخاصة على إتاحة فرصة مواصلة دراساتي العليا فيها .والشكر لكل من أعانني بنصح أو توجيه أو دعاء

أسأل الله تعالى أن يتقبل منا هذا العمل ، وأن ينفعنا بما علمنا ويزيدنا علماً إنه سميع مجيب .

وآخر دعوانا أن الحمد لله رب العالمين

((وصلى الله على النبي الأمين محمد وصحبه العز الميامين)) .

ملخص المقدمة

تشكل الظواهرغيرالخطية دوراً هاما في العديد من المسائل الفيزيائية وبصفة خاصة مســائل الموجات غير الخطية الــتي نقابلــها في كثير من المجالات مثل ميكانيكا السوائل، وفيزياء الحالة الصلبة، وفيزياء البلازما، والفيزياء الكيميائية.

وتمثل هذه المعادلات غير الخطية المطورة ومعادلات الموجات فئةً خاصة من المعادلات التفاضلية الجزئية التي تمت دراســتها بتوســع في العقود الماضية.حيث تستخدم هذه المعادلات لوصف معامل فيزيائي يوضح بعض أنواع خصــائص الانتشار أو التجميع. أحد الدوافع الفيزيائية الهامة في هذا البحث هو حل معادلات تفاضلية جزئية تتميــز بــأن لهــا حــل يظهـــر في صورة حل لموجه متحركة لها صفات خاصة بها تعرف بموجة السوليتون .(Solitary wave)

تتــناول هذه الأطروحة في الأساس دراسة الطرق العددية والتحليلية لحل مسألة ذات أهميه نظــريه وتطبيقيه وهي معادلة الكورتيج دي فريس المعممة (Generalized Korteweg De Vries Equation)

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{3x} = 0 \quad (I)$$



حــيث استخدمت بعض الطرق التحليلية المناسبة لإيجاد الحل المغلق لمعادلة كورتيج دي فريس المعممة وهي على النحو التالي:

١ – طريقة التكوين أو التكامل المباشر.
(A formal (direct integration) method)

٢– طريقة الحل غير المباشرة.
(Implicit method)

٣– طريقة دالة الظل الزائديه.
(Tanh (Hyperbolic tangent) method)

٤ – طريقة الأس الحقيقي أو متسلسلة القوى.
(Real exponential (Series expansion) method)

كما تم تطوير الطريقتين العدديتين التاليتين لمناقشة المسألة المدروسة.

١ – الطريقة الأولى هي طريقة زابوسكي وكرايسكال (Zabusky & Kruskal method)

٢ – الطريقة الثانية هي طريقة جريج وموريس (Hopscotch method)

اشتملت هذه الرســالة على ثلاثة فصول وثلاثة ملاحق وملخص باللــغة الإنجــليزية وأخر باللــغة العربية وقائمة بالمراجع التي احتوت على أهم الإنجازات في هذا المجال البحــثــي.

١) الفصل الأول: بعض الطرق التحليلية المستخدمة لحل معادلة الكورتــيج دي فريس المعممة

SOME ANALYTICAL METHODS FOR SOLVING THE GENERALIZED KORTEWEG DE VRIES EQUATION

اشتمل هذا الفصل على مــقدمة مــوجزة عن موضوع الرســالة، قدم فيها عــرض تاريخي لاكتشاف موجة السوليتون، وتوضيح علاقة هذه الموجات بحلول المعادلات التفاضلية الجزئية غير الخطية (NPDEs).كما أنها احتوت على أهم تطبيقات معادلة الكورتــيــج دي فريس المعممة في مجالات العلوم المختلفة.

وامـــتد هـــذا الفصـــل ليشمل شرح بعض الطرق التحليلية المتقدمة لإيجاد حلول موجة السوليتون للمعادلة المدروسة وهي الطرق الثلاثة الأولى آنفة الذكر، حيث قدم تحليلاً رياضياً دقيقاً لكل طريقة لاستخدامها في حل معادلة الكورتيج دي فريس((Kdv(p=1) ومعادلة الـــكورتيج دي فريس المعدلـــة ((Mkdv (p=2) ومن ثم تطـــبيقها على المعادلة المعـــممة بأســـاليب سهلة مبسطه.

وقد ارتكزت هذه الطرق على فرضية واحدة لحل المعادلات المطورة غير الخطية أو معادلات الموجـــة والتي تبدأ بالـــبحث عـــن حـــل الـــموجة المتحركة. وعلى هذا الأســـاس يمكن تحويل المعـــادلات غير الخطية إلى معادلات عادية غير خطيه. وبمقارنة الحلول الناتـــجة عن الطرق المتبعة في حل المعادلـــة المدروسة تبين أنها متفقة في الشكل النهائي للحـــل مـــع ما نشر في دراسات Hereman عام١٩٨٦م.

٢)الفصل الثاني بعنوان: طريقة الأس الحقيقي أو متسلسلة القوى

Real exponential (Series expansion) method

انفـــرد هـــذا الفصل بدراسة إحدى الطرق التحليلية الهامة المشهورة، التي استخدمت في بنـــاء حلول موجة السوليتون (SolitaryWave) للعديد من أنواع المعادلات غير الخطية المطورة ومعادلة الموجة. هذه الطريقة تعرف بطريقة الأس الحقيقي أو متســـلسلة القوى. ويعد العـــمل الـــذي نشـــر بواســـطة العالم Korpel عام ١٩٧٨ هو الأساس لهذه الطريقة.

اعتمدت هذه الطريقة على الحقيـــقة الرياضية القائلة بأن خلط الأسس الحقيقية أبسط كثيراً من بناء الدوال التوافقية مـــن الأس التخيلي بالنسبة لحلول النظام الخطي.

في بداية هذا الفصل تم تلخـــيص أسلوب الحـــل لهذه الطريقة في خطوات رئيســـية على المعادلات التفاضـــلية الجزئية غير الخطية تمهـــيداً لتطبيقـــها على المسألـــة المدروسة، حيث وصفت هذه الطريقة في دراســـات العـــالم Hereman عام١٩٨٦م. وقد اتسع هذا الفصل ليضم قســـماً لتطبيق هذه الإجراءات علـــى المعادلة المدروســـة، باختيار قيمتي ثابت التكامل وثابت المتسلسلة بحيث يساويان الصـــفر والذي بدأ بتطبـــيق هـــذه الطريقة على الحالات الأولى للمعادلة المدروسة. أي عند p=1,p=2,p=3,p=4,p=5,p=6 للوصول إلي الصورة الـــعامة للحل الحـــقيقي للمـــعادلة المعممه (G-KDV)، والذي يمكن منه استنتاج الحل الحقيقي للحالات السابقـــة التي رسمت باستخدام برنامج الرسم الخاص (Mathematica 3.0 program).

وفي نهاية هذا الفصل تم تطبيق خطوات هذه الطريقة على حالات خاصـــــــة مـــــن هذه المعادلـــة المعممة تعتمد على اختيار حالتين من الفرضيات بالنسبة لقيمتي ثابت التكامل وثابت المتسلسلة، والتي طـــبقت على معادلتي الكورتيج دي فريس ((Kdv (p=1) والكورتيج دي فريس المعدلة ((Mkdv (p=2) لينـــتج عـــنها الحل العام الذي يتوافق مع دراســـات العالم Hereman عام ١٩٨٦م على هذه المعادلات في حالة اختيار معامل الحد غير الخطي في المعادلة (I) يساوي ثابت.

٣)الفصل الثالث بعنوان : بعض طـــرق الفروق المنتهية لمعادلة الكورتــيج دي فريس المعممة

SOME OF FINITE DIFFERENCE METHODS FOR SOLVINGTHE GENERALIZED KDV EQUATION

إن الهدف من هذا الفصل هو دراسة وتطوير بعض الطرق العدديـــــة والمستخدمة لحـــل معادلات الموجات غير الخطية. ويمكن اختبار هذه الطرق على المعادلات التي عرف لها الحل الحقيقي والنتيجة التحليلية مسبقاً.

يحتوي هذا الفصل على مقدمة موجزة للتعريف بالطرق العددية المستخدمة لحل المعادلات غير الخطية الجزئية، وذكر التنظيم المتبع لمناقشة هذه الطـــــــرق الواردة في ثنايا أجزائه.

وفي القســـم الثاني من هذا الفصل تم تطوير إحدى الطرق العددية الأكثر شـــيوعاً لحل معادلـــة الكورتيج دي فـــريس المعممـــة وهـــي طريقة الفروق المنتهية حيث وصف نظامان لطريقة الفروق المنتهية (Finite Difference Method)

فالأول منها: طريقة زابوسكي وكرايسكال التقليدية (Zabusky & Kruskal method)

والثاني هو طريقة جريج وموريس (Hopscotch method).

بالإضافة إلى دراسة بعض العوامل المتعلقة بهذه الطريـــقة مثل خطأ القطع (Truncation error) وشرط الاستقرار (Stability condition).كما تم تخصيص الجزء الأخير من هذا الفصل لعرض ملخصات نتائج الطــــرق العددية التي درست على هذه المعادلة حيث صـــمم برنامج فورتران على الحاســــــــب الآلي لحل تلك المعادلة باستخدام الطريقـــتين المذكورتين سابقاً. كما استخدم برنامج الرسم (Mathematica 3.0 program) لإظهار شكل النتائـــــــــج العددية لكل طريقة عند قيم مختلفة في السرعة وزمن محدد للحصول على نتائج دقيقة ومتقاربة من الحل الحقيقي.

ولقد أتبعت في تسلسل كامل صفحات الرسالة طريقة طباعة الأبحاث العلمية المعتادة ، فجعلت الرقم في أسفل الصفحة ، ثم أعقبت ذلك بموجز أحتوى على أهم النتائج التي توصل إليها البحث ويلي ذلك قائمة بالمراجع ،ثم الملحقات التي انتهى الترقيم بها وذلك تمهيدا لنشرها إن شاء الله . وبعد فإن هذه محاولة متواضعة أردت أن أشارك بها في ميدان البحث العلمي خدمة للدراسات العلمية ، ولا سيما الرياضيات ، وما ابرىء نفسي من القصور أو التقصير فتلك طبيعة الإنسان ، فالسعي للكمال أمر محمود ولكن بلوغه أمر متعذر فالكمال لله وحده ، أما أعمال الإنسان فهي عرضة للخطأ والنسيان ويعلم الله أنني لم أدخر فيه جهدا ، ولا وقتا، ولا مالا ولم أبخل بسفر ، ولا سهر ، ولا دأب و لم أتردد في أي أمر يعود علي بالنفع فيه . فإن أك وفقت فهذا فضل من الله علي ومنّة منه ، جل وعلا ، وإن تكن الأخرى فحسبي أنني أخلصت النية، وبذلت ما أمكنني من جهد.

((وصلى الله على النبي الأمين محمد وصحبه العز الميامين))

اسم الباحثة | المشرف على الرسالة

أ/ شريفة عيسى علي الحازمي | د/كمال محمد عبدالله حميده

عميد الكلية

د / أحمد خماش

```
      ----------------------------------------------------------------
      SUBROUTINE TO SOLVE THE TRIDIOGNAL MATRIX

      SUBROUTINE TRID(SUB,DIAG,SUP,B,N)
      INTEGER L,K,J,LP1,LM1,JP1
      REAL SUP,SUB,DIAG,RATIO
      DIMENSION SUP(150),DIAG(150),SUB(150),B(150)
      ----------------------------------------------------------------

       DO 33,L=2,N
         LP1=L+1
         LM1=L-1
         RATIO=-SUB(L)/DIAG(LM1)
         DIAG(L)=RATIO*SUP(LM1)+DIAG(L)
         B(L)=B(L)+RATIO*B(LM1)
33     CONTINUE

      B(N)=B(N)/DIAG(N)
      L=N
       DO 34,K=2,N
         J=1+N-K
         JP1=J+1
         B(J)=(B(J)-SUP(J)*B(JP1))/DIAG(J)
34    CONTINUE
      RETURN
      END

      SUBROUTINE TO CALCULATE SECH^2
      ------------------------------
      FUNCTION SECHS(X)
      REAL A3,A4
      A3=COSH(X)
      A4=1/A3
      SECHS=A4*A4
      RETURN
      END
```

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Ministry of Higher Education
Umm Al-Qura University
Faculty of Applied Sciences
Department of Mathematical Sciences

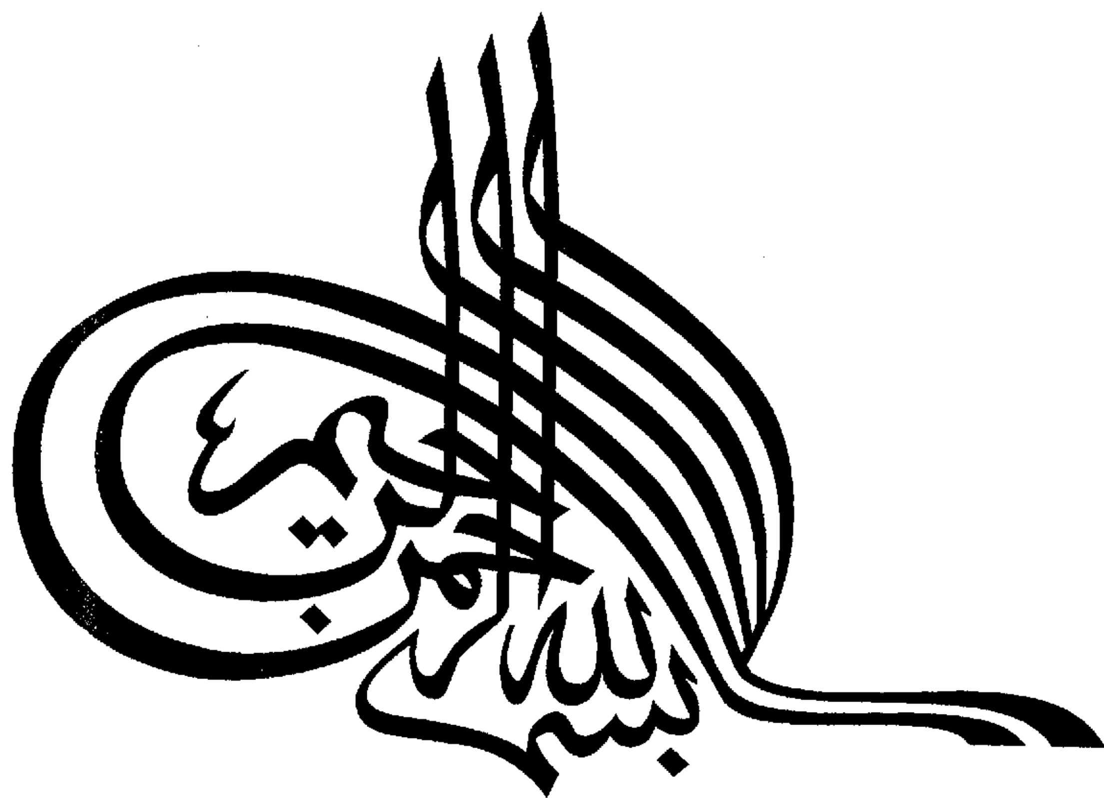
Theoretical and Numerical Studies of the Generalized Korteweg-De Vries Equation in Solitons

Prepared by
Shareefa Eisa Ali Al-Hazmi

Supervised by
Dr. Kamal Mohammed Abdullah Hemida

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بسم الله الرحمن الرحيم

CONTENTS

CHAPTER 3

SOME OF FINITE DIFFERENCE METHODS FOR SOLVING THE GENERALIZED KDV EQUATION

Summary

The nonlinear phenomena plays an important part in many physical problems and in particular, nonlinear waves are encountered in numerous domains such as fluid mechanics, solid-state physics, plasma physics and chemical physics.

Nonlinear evolution and wave equations are form a special class of partial differential equations, which have been studied intensively for past decades [1]. When a nonlinear partial differential equation is used to describe a physical parameter, which shows some kinds of propagation or aggregation properties, one of the important physical motivations is to solve the partial differential equations with the travelling (and / or solitary) solution.

The present thesis is mainly concerned to solve a Generalized Korteweg De-Vries (Generalized Kdv) equation according to Drazin [1] and Lax [2], which takes the form

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{3x} = 0,$$

where $u_{3x} = u_{xxx}$, γ is an arbitrary real number and p is almost a positive integer. The subscripts u_t and u_x refer to the partial derivatives of $u(x,t)$ with respect time (t) and space variable (x) respectively. For convenience, we use some of the analytical methods for finding the closed form-solution of the Generalized Kdv equation as

1. A formal (Direct integration) method [1].
2. Implicit method [3].
3. Tanh (Hyperbolic tangent) method [4].
4. Real exponential (Series expansion) method [5,6].

On the other hand, the following two numerical methods are developed for the studied problem

1. The first is Zabusky and Kruskal method [7].
2. The second is the Hopscotch method [8].

This thesis includes three chapters, three appendices and summaries in English and Arabic languages, as well as a list of references including the most important achievements in the considered field of research.

In chapter (1), various analytical methods are presented toward finding solitary wave solutions of the nonlinear wave equation, these methods are; A formal (Direct Integration) method [1], Implicit method [3] and Hyperbolic tangent (Tanh) method [4].

In the present chapter, we exemplify these techniques some times employing them to the Kdv and Mkdv equations and then applying them on the Generalized Kdv equation. Also, rigorous Mathematical analysis is given for each method used for solving the problem under studies (Generalized Kdv equation) in an easier and simpler ways.

Most of these techniques, however, are based on the fact that if one tries to solve nonlinear evolution and wave equations, one must first start to look for a travelling wave solution. In principle, these waves can be found easily, because the PDE (partial differential equation) under consideration can immediately be transformed into an ODE (ordinary differential equation).

Chapter (2) deals with one of the important well-known analytical approachs, which are used, in building up solitary wave solutions of several classes of nonlinear evolution and wave equations. This method is called the real exponential or a series expansion method [6]. The feasibility of this approach lies in the fact that, in a mathematical, it is far simpler to 'mix' real exponentials rather than harmonic functions built up from imaginary exponentials, incidentally, are also solutions of the linear system.

In the present chapter, we outlined the solution method in some steps, and we then applied the principal procedures of this technique on our problem (Generalized Kdv equation) for p = 1,2,3,4,5,6, and then three cases of the solutions for the Generalized Kdv equation are also discussed.

In the third chapter of the thesis we described some numerical methods for nonlinear wave equations with solitary wave solutions. These methods can be tested on equations for which analytic result and exact solutions are known.

Here, in this chapter, we began with a brief introduction to define some common numerical techniques and to give framework for the subsequent discussion. One of the most popular numerical techniques is the finite difference that was developed and applied on the Generalized Kdv equation, where we present two finite difference schemes for the numerical solution of the Generalized Kdv equation. The first is the classical Zabusky and Kruskal method [7] and the other is the Hopscotch method [8]. Some factors, which are connected with these methods (the finite difference) as well as the truncation error and the stability condition, are discussed.

The major part of this chapter is devoted to summaries of the results of numerical methods that are investigated on such equations. Two Fortran programs are also written and the numerical results are illustrated and plotted using (mathematica 3.0 program) [9] and acomparison between these methods is shown.

Chapter(1)

CHAPTER (1)

SOME ANALYTICAL METHODS FOR SOLVING THE GENERALIZED KORTEWEG DE VRIES EQUATION

1.1 INTRODUCTION

Several names like solitons, kinks, breathers, etc. are now commonly used in the vast literature [1] dealing with the study some analytical methods for solving the Generalized Korteweg De Vries equation. Therefore, simple techniques and methods are needed to investigate and study the nonlinear evolution and wave equations.

There are explicit stationary travelling wave solutions of nonlinear dispersive evolution and wave equations can be derived using a variety of well-known techniques. Notable among these are direct integration (wherever possible), the inverse scattering method (Ablowitz and Sequr) [10], the Bäckland transformation technique (Miura) [11], the Hirota method (Hirota)[12], perturbation techniques (Whitham) [13], the summation process of the pade'type (Liverani and Turchetti) [14], direct linearization technique (Santini *et al*) [15], real exponential approach ((Korpel) [16], (Hereman *et al*) [5,6]) and the Hyperbolic tangent method ((Hiubin) [17], (Malfliet) [4,18]).

The key to this present knowledge of these equations is the realization that they possess a special type of elementary solution. These special solutions take the form of localized disturbances, or pulses, that retain their shape even after interaction among themselves, and thus act somewhat like particles. This independence among elementary solutions is a well-known effect in processes governed by linear partial differential equations where a linear superposition principle applies but was quite unexpected when first observed in processes governed by nonlinear partial differential equations. These localized disturbances are known as solitons, where solitons are a special kind of localized wave, an essentially nonlinear kind.

The theory of solitons is attractive, it is wide and deep, and it is intrinsically beautiful. It is related to even more area of mathematics and has even more application to the physical sciences. It has an interesting history and a promising future.

1.1-1 <u>The Discovery of the Solitary Wave</u>

A common theme in the development of science is that of an important discovery, which is not widely recognized as such when it is first reported. A solitary wave is the first and most celebrated example of a soliton to have been discovered, although more than 150 years elapsed after the discovery before a solitary wave was recognized as an example of a soliton.

Most often this comes about not through the ignorance or indifference of the scientific community, but because the current state of knowledge of the field is insufficiently developed for the full significance of the result to be realized.

The first conscious observation of what was termed a solitary wave in 1834 was not appreciated until its significance as an important stable state of some nonlinear systems which was realized in the mid- 1960's. A solitary wave may be defined more generally than as a sech squared solutions of the Kdv equation, we take it to be any solution of a nonlinear system which represents a hump-shaped wave of permanent form. The solitary wave also has since appeared in many fields of applied mathematics and physics such as meteorology, elementary particle physics, plasma studies, and laser physics.

The first documented observation of the solitary wave was made in 1834 by the Scottish scientist and engineer, John Scott Russell [19]. Whilst observing the movement of a canal barge, Scott Russell noticed a novel type of water wave on the surface of the canal.

On a summer day in 1834 John Scott Russell become excited when he saw the emergence of a solitary wave from a stopped barge in the Edinburgh Glasgow canal. This elevated mass of water continued down the channel for a couple of miles without changing its shape. Russell reported his observations and experimental results to the

British Association for the advancement of science. However, his findings started a fifty years controversy among the leading experts, since they were firm in their beliefs that such localized waves would eventually disperse.

In 1895 Russell's "wave of translation" had finally found theoretical support. Two Dutchman, Korteweg and De Vries [20], had derived an equation, governing the propagation of shallow water waves. The Korteweg De Vries (Kdv) equation related the spatial changes in the amplitude of the wave to its temporal changes and possessed a solution, which matched Russell's observation. Interest in solitary waves diminished, once their existence was confirmed. Russell also did some laboratory experiments, generating solitary wave by dropping a weight at one end of water channel (see Fig (1)). Where he deduced that the steady velocity c of the wave is given by

$$c^2 = g\,(h + a)\,,$$

where a is the amplitude wave and h is the highest of the undisturbed water and the volume of the water in the wave is equal to the volume displaced by the weight , also, he tried to generate waves of desperation by raising the weight from the bottom of the channel, initially. He found, however, that an initial depression becomes a train of an oscillatory waves whose length increase and amplitudes decrease with time (see Fig (2))

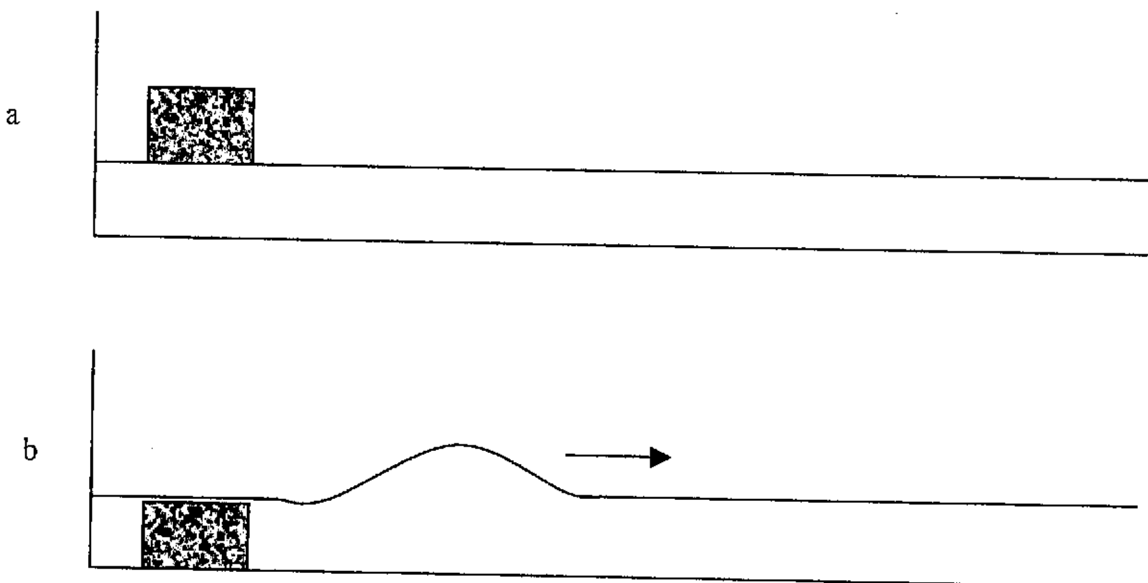


Fig (1) Russell'solitary wave: a diagram of its development. (a) The start. (b) Later (After Russell 1844)

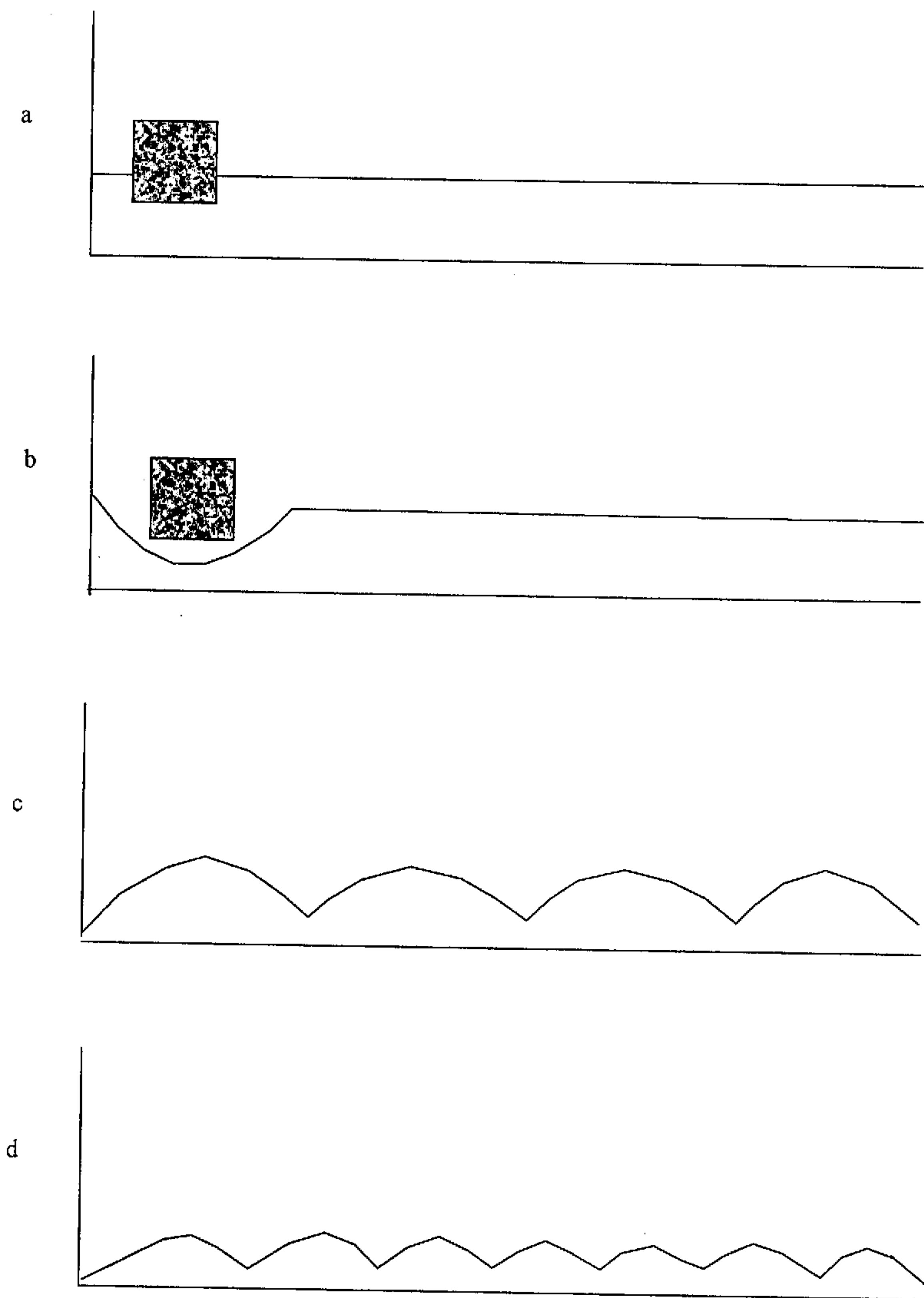


Fig (2) Russell's observations of oscillating waves: successive stages of their development. (After Russell 1844).

Boussinesque [21] and Rayleigh [22] independently assumed that a solitary wave has a length much greater than the depth of the water. They further showed essentially that the wave height above the mean level h is given by

$$\xi(x,t)=a\,\mathrm{sech}^2\left((x-ct)/b\right),$$

where $b^2 = 4h^2(h+a)/3a$ for any positive amplitude a.

In (1895) Korteweg and De-Vries [20] developed this theory, and found an equation governing the two-dimensional motion of weakly nonlinear long waves:

$$\frac{\partial \xi}{\partial t}=\frac{3}{2}\sqrt{\frac{g}{h}}\left(\xi\frac{\partial \xi}{\partial x}+\frac{2}{3}\alpha\frac{\partial \xi}{\partial x}+\frac{1}{3}\sigma\frac{\partial^3 \xi}{\partial x^3}\right),\quad \alpha,\sigma \text{ constants.}$$

This is essentially the original form of the Kdv equation. Note that in the approximation used to derive this equation one consider long waves propagating in the direction of increasing x.

1.1-2 <u>The Discovery of Soliton Interaction</u>

Up until this point we have talked about solitary wave solutions which, using a very lose definition, are no more than waves which propagate without change of form and have some localized shape, where the solitary wave, or great wave of translation was first observed on the Edinburgh to Glasgow canal in 1834 by J. Scott Russell [19]. Scott Russell' s particular interest was in solitary waves in shallow water.

The word "soliton" first appears in the work of Zabusky and Kruskal [7]. Kruskal had been interested for some time in the FPU (Fermi - Pasta -Ulam) problem and particularly why recurrence occurred. Zabusky and Kruskal describe a numerical study of the Kdv equation with a factor δ^2 multiplying the $\frac{\partial^3 u}{\partial x^3}$ term, they choose $\delta = 0.022$, periodic boundary conditions such that $u(x,t) = u(x+2,t)$, and a periodic initial condition u (x, 0) = cos x. Zabusky and Kruskal computed u for t > 0. They found that the solution breaks up into a train of eight solitary wave, each one like a sech-

squared solution, that these waves moves through one another as the faster ones catch up the slower ones, and that finally the initial state (or some thing very close to it) recur. This remarkable numerical discovery, that strongly nonlinear waves can interact and carry on thereafter almost as if they had ever interacted, led to an intense study of the analytical and numerical properties of many kinds of solitons.

In 1965 the Kdv equation became the focus of attention again, but in a different area. Zabusky and Kruskal were studying the heat conductivity of solids through numerical simulations of the Kdv equation. They observed that a simple initial profile broke up into a train of solitary waves with the tallest ones moving the fastest. When Zabuzky and Kruskal allowed two of these pulses to collide, they found that the pulses passed through each other, coming out of the interaction without any change in shape. Since these pulses behaved more like particles than waves, they were given the name solitons. This is happens, in which case two solitons of different amplitude are moving in the same direction and the taller wave eventually overtake the shorter one. By arranging the tracing of the waves such that the vertical axis is time, one notices that the taller wave has been pushed slightly forward and the shorter wave has been kicked backwards due to the collision.

One of the most promising applications of soliton theory is in the field of optical communications. In these systems information is encoded into lights pulses and transmitted through optical fibers over large distances. Commercial systems have been in operation since 1977 and a transatlantic undersea optical cable has been developed, which is expected to transmit around 40.000 telephone conversations simultaneously.

In 1973 Hasegawa and Tappert [23] had proposed that soliton pulses could be used in optical communications. However, the technology was not available until seven years later, at which time researchers at Bell laboratories had experimentally demonstrated the propagation of solitons in optical fibers.

Real soliton pulses can become distorted as a result of impurities and in homogeneities in the fiber, causing the solitons to broaden. After some time, neighboring solitons would overlap, resulting in a loss of information.

At the end we can say that a soliton is not precisely defined, but is used to describe any solution of a nonlinear equation or system which

- Represents wave of permanent form
- Is localized, decaying or becoming constant at infinity
- May interact strongly with other solutions so that after the interaction it retains its form, almost as if the principle of superposition was valid.

1.1-3 The Previous Studies of the Kdv and MKdv Equations

In this thesis, we will study one of the most important well known nonlinear wave equation, this equation is called the Generalized Kdv equation [1,2]. Specially, the Korteweg De Vries (Kdv) equation and The Modified Korteweg De Vries (Mkdv) equation represent the common and important special cases of the Generalized Kdv equation, since they have number of the properties and applications to selected cases.

The intense research activity of the past years surrounding the Kortewag De-Vries (Kdv) equation was motivated by Miura and Gardner [11], they derived a variety of conservation laws and constants of motion for the Kdv equation. And the sturm – lioville eigen value problem is explained and applied in Kdv equation.

Then, some methods for find exact solution of the Kdv equation illustrated by Gardner and co-workers [24]. They give more details to the inverse scattering theory. Also, the case of pure soliton solutions is discussed, and the number of results involving the eigen-values and corresponding eigen-function of schrödinger equation are given.

Therefore, a general method for finding evolution equation having infinitely many symmetries or flows, described by (Olver) [25], and applied it for the Kdv equation, Mkdv equation, Burger's equation, and Sine-Gorden equation. This method depends on an infinite series of conservation laws, and the connection between one-parameter symmetry groups of the equation and conservation laws.

Using the Kdv equation as an example, a model of solution formation is demons -treated by Korpel [16] in which the nonlinearty mediates the coupling between real exponential waves, characteristic of the linear medium. It is purpose of their note to demonstrate an alternative, and intuitively perhaps more simple, synthesis which uses the real, rather than the complex, exponential waves characteristic of the linear medium. Such waves are of the form $\exp(\pm\beta(u)(x-ut))$ where $\beta(u)$ is a real function of u.

Then, the direct linearization (DL) introduced by Santini *et al* [15], in connection with the Korteweg-De Vries (Kdv) equation. It is based essentially on the existence of an integral equation, this approach also used to generate particular solutions of nonlinear evolution equations that can be solved by the inverse scattering transform.

Hereman *et al* [5,6] are then introduced an algebra (real exponential or a series expansion) approach to obtaining the travelling wave solutions (solitary) of kink type for several classes of nonlinear partial differential equations as the Mkdv equation, Kdv equation with additional fifth order dispersion term (see chapter (2)).

In (1990), Coffey [26] developed a series expansion technique which is using by Hereman *et al* [6] to found kink and antikink solutions for Kdv- like equation with fifth-degree non-linearity and a combined Kdv and Mkdv equation for restricted values of the coefficients of non-linear terms, where their solutions are in agreement with those found by Dey [27].

Next, he essential procedures of the implicit solitary wave solutions of nonlinear PDEs are summarized by (Banerjee *et al*)[3]. This method is applied for finding implicit solutions to the Kdv, Mkdv and Generalized Kdv equations, where the implicit solutions that derived in their paper are new and not previously known, because the

solutions are obtainable only by choosing D/H_1 as a function of f rather than F. The figures of the result solutions are also different from the conventional sech-or $sech^2$-type solutions (see section (1.3)).

Quite recently, travelling wave solutions of the Kdv and Mkdv equations found in Malfliet [4] by using the Hyperbolic tangent (Tanh) method, where it is based on the fact that the most solutions of nonlinear wave equations are functions of a tanh. This technique is a straightforward to use and only minimal algebra is needed to find those solutions (see section (1.4)).

Thereafter, Yang [28] has studied a series of a new ansätze, which can be related to some nonlinear evolution / wave equations in more general sense, this method are applied for solve several classes of nonlinear partial differential equations as the Generalized Kdv equation, generalized Burger's equation and Modified Sine-Gordon equation, etc. In their paper also, the solutions for various kinds of the NPDEs can be expressed in a hyperbolic function as cosecant or tangent or cotan, etc, where the present ansätze approach only involves algebraic calculations.

1.1-4 <u>Application of the Generalized Kdv Equation in Solitons</u>

Here, we will speak of some applications of the Generalized Kdv equation

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{xxx} = 0,$$

where γ is an arbitrary real number and p is almost a positive integer, this equation describes abroad spectrum of physical phenomena for systems in which small oscillations have an caustic spectrum, i.e. w (k) $\rightarrow$k at k$\rightarrow$0. However, it has an unpleasant feature.

For p = 1, the celebrated Kdv equation (Korteweg and De Vries) [20], (Lamb) [29] and (Hereman *et al*) [6] which is recognized as the prototype of modern nonlinear wave theory, is written as

$$u_t + 6uu_x + u_{3x} = 0.$$

The Kdv equation introduced by Korteweg and De Vries [20] was originally derived in order to describe the behavior of one-dimensional shallow water with finite

amplitudes. While the Kdv equation is first studied numerically by Zabusky and Kruskal [7].

It plays an important role in plasma physics (ion-acoustic waves for instance) (Infeld) [30] and in solid-state physics; also it has been found to describe a harmonic crystal, bubble liquid mixture. More generally, (Su and Gardner) derived it for wide classes of nonlinear Galilean-invariant systems where dispersion is dominant and the long-wavelength approximation is used.

The Kdv equation possesses steady progressing wave solutions. These are in the form of either uniform wave trains or solitary waves. Of more interest to us here are the solitary wave, which are given explicitly by

$$u(x,t) = 3a^2 \operatorname{sech}^2\left[\frac{1}{2} a\left(x - x_0 - a^2 t\right)\right],$$

if it is assumed that $u \to 0$ at $x \to \pm\infty$. These waves travel to the right at a constant speed a^2 that is proportional to their amplitude $3a^2$. The width of the wave is inversely proportional to the square root of the speed.

On the other hand, the second case for $p = 2$ (Mkdv equation) (Lamb) [29], (Dodd *et al*) [31], which is quite similar to the Kdv equation but has a cubic non-linearity. Both equations are connected by the Miura transformation (Lamb) [29] where the Modified kdv equation is written as

$$u_t + 12u^2 u_x + u_{3x} = 0.$$

For instance, This equation describes a wide class of physical phenomena as acoustic waves in certain an harmonic lattices (Zabusky) [32] and Alfen waves in a collisionless plasma, and it appears in electric circuit theory, double layer theory, and serves as a model of solitons in a multi-component plasma and phonons in an harmonic lattice [30].

For the Generalized Kdv equation if $p = 3$ is describing the propagation of ion-acoustic waves at critical densities in multi-component plasma with different ionic

charges and temperatures. Also, for $p = \frac{1}{2}$ the equation describing ion-acoustic waves in a cold-ion plasma but where electron don't behave isothermally during their passage of the wave [28].

The plane of the work in this chapter is

In this chapter, various analytical methods are presented toward finding solitary wave solutions of the nonlinear wave equation.

The first method, which treats in section (1.2), is a formal (Direct integration) method. This method used earlier by Drazin [1] to constructing the single solitary wave solution for the Kdv equation. The possibility of using this technique in such cases depending on some simple integration and considerations, so the exact solutions of such type are derived with straightforward and (in most cases) simple algebra. Here, this method can be extended to construct the solitary wave solution, taking the Generalized Kdv equation as example and then apply it on the Kdv and Mkdv equations to find the final form of the exact solutions for them.

In section (1.3), we will investigate the possibility of structure implicit solitary wave solutions to some integrable PDEs, e.g., Kdv, Mkdv and the Generalized Kdv equations. The outline of the solution method of the implicit method [3] are summarized and applied to find the implicit solution for the previous equations above.

Brief discussions of the nature of theses solutions are given, however, the solutions of this approach are inherently implicit and different from both the well-known explicit solutions derivable from the classical method. Here, new closed form solutions are obtainable for $p = 1$ (Kdv equation), $p = 2$ (Mkdv equation) and $p = 4$ (Generalized Kdv equation), again, for $p = 3$ and $p > 4$ no closed form solutions appear.

As an alternative, in section (1.4), the Hyperbolic tangent (Tanh) method [4] is introduced to solve particular nonlinear evolution and wave equations. With the aid of the Tanh method, we have derived travelling-wave solutions of solitary waveform using

relatively simple mathematics. In this section, we outline how this method can be applied for solving the equations under study (the Kdv, Mkdv, and the Generalized Kdv equations), and exact closed from solutions can be obtained easily for this cases.

The main feature of this approach is based on the hypothesis that the travelling wave solutions may be expressed as a power series in tanh. This hyperbolic function is then used as independent variable. As a result, algebraic equations appear from which the coefficients of the power series are determine, and a straightforward analysis can then be carried out so that the method will be applicable to large class of equations.

1.2 A FORMAL (DIRECT INTEGRATION) METHOD

Consider the general form of the Generalized Kdv equation

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{3x} = 0, \qquad (1.2.1)$$

where $u_{3x} = u_{xxx}$, γ is an arbitrary real number and p is almost a positive integer. If we take p = 1, we have the usual form

$$u_t + 6u\, u_x + \gamma u_{3x} = 0, \qquad (1.2.2)$$

and if p = 2, equation (1.2.1) rewrites as

$$u_t + 12u^2 u_x + \gamma u_{3x} = 0. \qquad (1.2.3)$$

To solve it, first seek waves of permanent shape and size by trying solutions such that

$$u(x,t) = f(X), \qquad X = x - vt, \qquad (1.2.4)$$

for some function f and constant wave velocity v.

Substituting from (1.2.4) into (1.2.1) and integration the result equation, we have

$$\gamma f_{XX} = vf - (p+2) f^{p+1} + A. \qquad (1.2.5)$$

Multiplying (1.2.5) by f_X and integrating, we have

$$\gamma f_X^2 = vf^2 - 2f^{p+2} + 2Af + B, \qquad (1.2.6)$$

where $f, f_X, f_{XX} \to 0$ as $X \to \pm\infty$. Then, A and B, constants of integration, vanish. i.e.

$$\gamma f_X^2 = vf^2 - 2f^{p+2}, \qquad (1.2.7)$$

or

$$f_X = f\left(\frac{v-2f^p}{\gamma}\right)^{\frac{1}{2}}.$$

Using $f^p = \frac{1}{2} v \operatorname{sech}^2 z$, separating variables and integrating, one obtains

$$z = -\frac{1}{2} p \left(\frac{v}{\gamma}\right)^{\frac{1}{2}} (X - x_0),$$

and

$$f^p = \frac{1}{2} v \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{v}{\gamma}}\,(X - x_0)\right),$$

where x_0 is a constant of integration. Consequently,

$$u^p = \frac{1}{2} v \operatorname{sech}^2\left(\frac{p}{2}\sqrt{\frac{v}{\gamma}}\,(x - vt - x_0)\right). \tag{1.2.8}$$

Then, the solution of (1.2.2) and (1.2.3) are

$$u = \frac{1}{2} v \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{v}{\gamma}}\,(x - vt - x_0)\right), \tag{1.2.9}$$

and

$$u = \left(\frac{v}{2}\right)^{\frac{1}{2}} \operatorname{sech}\left(\sqrt{\frac{v}{\gamma}}\,(x - vt - x_0)\right). \tag{1.2.10}$$

Equation (1.2.9) describes a soliton with amplitude $\frac{1}{2}v$, which is proportional to its velocity. Hence, a teller soliton moves faster than a smaller one. The width is proportional to $(v)^{\frac{1}{2}}$ and a constant x_0 plays the role of a phase or center shift.

If the coefficient of the nonlinear term in equation (1.2.1) has a negative sign and p is odd then the solution is negative, i. e.

$$u^p = -\frac{1}{2} v \operatorname{sech}^2\left(\frac{1}{2} p\sqrt{\frac{v}{\gamma}}\,(x - vt - x_0)\right), \tag{1.2.11}$$

but if p is even, the solution is not a solitary wave [1].

1.3 THE IMPLICIT METHOD

The purpose of this section is to demonstrate the applicability of the implicit method [3] to solve the nonlinear PDEs. Selected, in this part, we will investigate the possibility of constructing the implicit solitary wave solution of the Generalized Kdv equation (1.2.1), using the principal procedures, which are summarized by [3].

The implicit solutions, which are derived in this section for the Kdv, Mkdv and Generalized Kdv equations, are not in the standard solutions. Furthermore, conventional solutions of the Kdv and Mkdv equations are obtainable by this method are depend on the choosing $\frac{D}{H_1}$ as a function of f rather than F. Moreover, the solutions of the above equations, when plotted [3], are cusp-type and different from the conventional sech-or $sech^2$ -type solutions.

1. 3 -1 The Solution Method

The principal technique for finding the implicit solitary wave solution of non-linear PDEs will be illustrating in the following steps.

1- First, the solution of the nonlinear PDEs is expressed in the implicit form

$$u(x,t) = F(f(x,t)), \qquad (1.3.1)$$

with

$$f(x,t) = H_1(f)\,x - H_2(f)\,t + H_3(f), \qquad (1.3.2)$$

where f (x,t) may be regarded as a Riemann invariant while the implicit solution u is what has been known as the Riemann wave (Whitham) [13].

2- With this assumptions, the given non-linear PDEs in u (x, t) transform into an ODE in $F(f)$. The coefficients in the ODE will include H_1, H_2 and H_3 and their derivatives with respect to f, where the derivatives in the (NPDEs) is changed as

$$\frac{\partial u}{\partial t}=\frac{dF}{df}\frac{\partial f}{\partial t}, \qquad \frac{\partial u}{\partial x}=\frac{dF}{df}\frac{\partial f}{\partial x},$$

and subsequently calculating f_t and f_x from (1.3.2) to give

$$f_t=\frac{-H_2}{D}, \qquad f_x=\frac{H_1}{D}, \tag{1.3.3}$$

with

$$D(f)=1-H_{1,f}x+H_{2,f}t-H_{3,f}, \tag{1.3.4}$$

then

$$u_t=-\frac{H_2}{D}F_f, \qquad u_x=\frac{H_1}{D}F_f. \tag{1.3.5}$$

Since we are only interested in stationary wave solution that does not change their shape, we have to set

$$H_2=v\,H_1, \tag{1.3.6}$$

where v is the velocity of travelling wave. Hence, we can define

$$\left.\begin{aligned}
u_t &= \frac{-vH_1}{D}F_f, \qquad u_x=\frac{H_1}{D}F_f,\\
u_{2x} &= \frac{d}{df}\left(\frac{H_1}{D}F_f\right)\frac{\partial f}{\partial x} = \frac{H_1}{D}\frac{d}{df}\left(\frac{H_1}{D}F_f\right),\\
u_{3x} &= \frac{d}{df}\left(\frac{H_1}{D}\frac{d}{df}\left(\frac{H_1}{D}F_f\right)\right)\frac{\partial f}{\partial x} = \frac{H_1}{D}\frac{d}{df}\left(\frac{H_1}{D}\frac{d}{df}\left(\frac{H_1}{D}F_f\right)\right).
\end{aligned}\right\} \tag{1.3.7}$$

3- Then, we integrate the resulting equation until we find the solution for F in terms of f. From the problem in the following section, we final step usually entail an expression of the form

$$\frac{dF}{df}=\left(\frac{D}{H_1}\right)F^{\mu}\{P(F)\}^{\frac{1}{2}}, \tag{1.3.8}$$

where μ is a constant and P (F) is a polynomial in F.

4- Now, the crucial step for finding implicit solution is depended on choosing $\frac{D}{H_1}$ to be an appropriate explicit function of F rather than f, such that the given form in (1.3.8) is integrably, and with proper choices for some of the integration constants.

5- Thereafter, the relation between the implicit variable f with x and t will be determine as:

From (1.3.4) and (1.3.6), we get

$$x - vt = \frac{(1 - D - H_{3,f})}{H_{1,f}}. \tag{1.3.9}$$

6- To find the general form of angle phase H_3, we substitute in (1.3.2) from (1.3.6) to get

$$f = H_1(x - vt) + H_3 \ . \tag{1.3.10}$$

Using (1.3.9) into (1.3.10), we have

$$f = H_1 \frac{(1 - D - H_{3,f})}{H_{1,f}} + H_3,$$

or

$$H_{3,f} - (H_{1,f}/H_1)H_3 = 1 - (H_{1,f}/H_1)f - D, \tag{1.3.11}$$

which, upon division by H_1, may be integrated to give

$$H_3(f) = f - H_1(f)\int \left(D/_{H_1}\right)df + CH_1(f)\ , \tag{1.3.12}$$

where C is the integration constant. By choosing an appropriate function of f for H_1, we can solve (1.3.12) for H_3.

7- After performing the implicit solution (F and H_3 as function of f) and f as a function of x and t in (1.3.2), we can plot the explicit solution u (x,t) against x and t.

1. 3- 2 The Generalized Korteweg De Vries Equation

In this subsection, we are applying the outline of the solution method as above to find the implicit solution of the Generalized Kdv equation (1.2.1) [1,2]. Specifically, closed-from solutions are obtainable only for the cases p = 1 (Kdv equation), p = 2 (Mkdv equation) and p = 4 (the Generalized Kdv equation).

In accordance with steps (1), (2), we first rewrite (1.2.1) entirely in terms of f as:

$$-vF_f+(p+1)(p+2)F^pF_f+\gamma\left(\frac{d}{df}\right)\left(\frac{H_1}{D}\frac{d}{df}\left(\frac{H_1}{D}F_f\right)\right)=0, \qquad (1.3.13)$$

where u and its derivatives are expressible as:

$$\left.\begin{aligned}
&u(x,t)\to F(f),\\
&u_t\to -v\left(\frac{H_1}{D}\right)F_f,\\
&u_x\to \left(H_1/D\right)F_f,\\
&u_{3x}\to \left(H_1/D\right)\left(\frac{d}{df}\right)\left[\left(H_1/D\right)\left(\frac{d}{df}\right)\left(\left(\frac{H_1}{D}\right)F_f\right)\right].
\end{aligned}\right\} \qquad (1.3.14)$$

Integrating one yields

$$-vF+(p+2)F^{p+1}+\gamma\frac{H_1}{D}\left(\frac{d}{df}\right)\left(H_1/D\,F_f\right)=c_1. \qquad (1.3.15)$$

Multiplying by F_f and integrating, equation (1.3.15) gives

$$df=\left(\frac{D}{H_1}\right)^{-1}\left(\frac{v}{\gamma}F^2-\frac{2}{\gamma}F^{p+2}+\frac{2c_1}{\gamma}F+\frac{2}{\gamma}c_2\right)^{-\frac{1}{2}}dF\ , \qquad (1.3.16\text{-a})$$

and

$$f=\int\left(\frac{D}{H_1}\right)^{-1}\frac{dF}{\left(\frac{v}{\gamma}F^2-\frac{2}{\gamma}F^{p+2}+\frac{2c_1}{\gamma}F+\frac{2c_2}{\gamma}\right)^{\frac{1}{2}}}, \qquad (1.3.16\text{-b})$$

where c_1 and c_2 are the integration constants.

Now, the particular cases of p = 1 (Kdv), p = 2 (Mkdv) and p = 4 of the Generalized Kdv, which are known to be integrable, will be discussion as well as it showed in [3].

1. First, if we take p = 1 (Kdv equation), then the general form (1.2.1) changes to (1.2.2). Adhering to the strategy of the solution method above, equation (1.3.16-b) rewrites as

$$f=\int\left(\frac{D}{H_1}\right)^{-1}\left(-\frac{2}{\gamma}F^3+\frac{v}{\gamma}F^2+\frac{2c_1}{\gamma}F+\frac{2c_2}{\gamma}\right)^{-\frac{1}{2}}dF. \qquad (1.3.17)$$

For convenience, we choose $c_2=0$ and

$$\frac{D}{H_1}=(-F)^{\frac{1}{2}} \qquad (1.3.18)$$

Now, if we set $\frac{2}{\gamma}=c$, $\frac{v}{\gamma}=-b$ and $\frac{2c_1}{\gamma}=-a$, equation (1.3.17) reduces to

$$f=\int\frac{dF}{F(cF^2+bF+a)^{\frac{1}{2}}}. \qquad (1.3.19)$$

Suppose that $F=\frac{1}{z}$, then

$$-a^{\frac{1}{2}}f=\int\frac{dz}{\left(z^2+\frac{b}{a}z+\frac{c}{a}\right)^{\frac{1}{2}}}=\int\frac{dz}{\left(\left(z+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right)^{\frac{1}{2}}}.$$

Consequently,

$$-a^{\frac{1}{2}}f=\sinh^{-1}\frac{(2az+b)}{\sqrt{4ac-b^2}}. \qquad (1.3.20)$$

Since $z=\frac{1}{F}$, then

$$F=\frac{2a}{-\sqrt{4ac-b^2}\,\sinh(\rho)-b}, \qquad (1.3.21)$$

or

$$F=\frac{2a(-b+\sqrt{4ac-b^2}\ \sinh(\rho))}{b^2-(4ac-b^2)\sinh^2(\rho)}=\frac{2a(-b+\sqrt{4ac-b^2}\ \sinh(\rho))}{b^2\cosh^2(\rho)-4ac\sinh^2(\rho)}.$$

with $b^2 \le 4ac,\quad \rho=a^{\frac{1}{2}}f$. Consequently, one can easily deduce that

$$F(f)=\frac{-b+\left(\left(4ac-b^2\right)\dfrac{\tanh^2(\rho)}{1-\tanh^2(\rho)}\right)^{\frac{1}{2}}}{\left(b^2-4ac\tanh^2(\rho)\right)/2a\left(1-\tanh^2(\rho)\right)},\qquad (1.3.22)$$

with $b^2 \le 4ac,\quad \rho=a^{\frac{1}{2}}f$. Notice that, if we suppose that $F=-\dfrac{1}{z}$ and we substitute into (1.3.19), by the same analysis above, we have

$$F(f)=\frac{-b-\left(\left(4ac-b^2\right)\dfrac{\tanh^2(\rho)}{1-\tanh^2(\rho)}\right)^{\frac{1}{2}}}{\left(b^2-4ac\tanh^2(\rho)\right)/2a\left(1-\tanh^2(\rho)\right)},\qquad (1.3.23)$$

with $b^2 \le 4ac,\quad \rho=a^{\frac{1}{2}}f$. The equations (1.3.22) and (1.3.23) are agreeing with the result in [3]. i.e.

$$F(f)=\frac{-b\pm\left(\left(4ac-b^2\right)\dfrac{\tanh^2(\rho)}{1-\tanh^2(\rho)}\right)^{\frac{1}{2}}}{\left(b^2-4ac\tanh^2(\rho)\right)/2a\left(1-\tanh^2(\rho)\right)},\qquad b^2\le 4ac,\quad \rho=a^{\frac{1}{2}}f.$$

The fifth step involves the evaluation of $H_3(f)$. Using (1.3.18) into (1.3.12), we can write

$$H_3(f)=f-H_1\int(-F)^{\frac{1}{2}}df+CH_1.\qquad (1.3.24)$$

Then, we can determine F(f) and $H_3(f)$ as a function in f. In [3] they take $H_1=a=b=c=1$ and C = 0 to compute F and $H_3(f)$ against f. Consequently, they calculated $u(x,t)$ and $H_3(x,t)$ against x, when f is taken as a function of (x-vt) at t = 0

2. The second, to show the implementation of implicit solutions for $p = 2$, we have (Mkdv equation) (1.2.3). As in the Kdv equation case, equation (1.3.16) changes to

$$f = \int \left(D/_{H_1} \right)^{-1} \left(-\frac{2}{\gamma}F^4 + \frac{v}{\gamma}F^2 + \frac{2c_1}{\gamma}F + \frac{2c_2}{\gamma} \right)^{-\frac{1}{2}} dF. \qquad (1.3.25)$$

For simplicity the solution in this case we choose $c_1 = 0$ and introduce a new function G such that

$$F = (-G)^{\frac{1}{2}}, \qquad (1.3.26)$$

and select

$$\frac{D}{H_1} = \frac{F}{2}. \qquad (1.3.27)$$

With these considerations [3], equation (1.3.25) becomes

$$f = \int \frac{dG}{G\left(-\frac{2}{\gamma}G^2 - \frac{v}{\gamma}G + \frac{2}{\gamma}c_2 \right)^{\frac{1}{2}}}. \qquad (1.3.28)$$

By choosing $c = \frac{-2}{\gamma}$, $b = \frac{-v}{\gamma}$ and $a = \frac{2c_2}{\gamma} > 0$, equation (1.3.28) gives

$$f = \int \frac{dG}{G\left(cG^2 + bG + a \right)^{\frac{1}{2}}}. \qquad (1.3.29)$$

Following the same steps as in Kdv equation above, one can easily deduce

$$G = \frac{-b + \left(\left(4ac - b^2\right) \frac{\tanh^2(\rho)}{1 - \tanh^2(\rho)} \right)^{\frac{1}{2}}}{\left(b^2 - 4ac\tanh^2(\rho)\right) / 2a\left(1 - \tanh^2(\rho)\right)}, \qquad (1.3.30)$$

with $b^2 \leq 4ac$, $\rho = a^{\frac{1}{2}}f$. Then the corresponding solution to (1.3.13) for $p = 2$ is

$$F(f) = \left[\frac{b - \left(\left(4ac - b^2\right)\tanh^2(\rho) / \left(1 - \tanh^2(\rho)\right) \right)^{\frac{1}{2}}}{\left(b^2 - 4ac\ \tanh^2(\rho)\right) / 2a\ \left(1 - \tanh^2(\rho)\right)} \right]^{\frac{1}{2}}, \quad b^2 \leq 4ac, \quad \rho = a^{\frac{1}{2}}f, \qquad (1.3.31)$$

while H_3 from (1.3.12), upon taking $H_1(f) = f$ is

$$H_3(f) = (C+1)f - f\int \left(\frac{F}{2}\right) df. \tag{1.3.32}$$

Then, we can determine F and $H_3(f)$ as a function in f. In [3] they take $a = 3,\ b = 1,\ c = 0.25$ and $C = 1$ to compute F (f) and $H_3(f)$ against f. Consequently, they calculated $u(x,t)$ and $H_3(x,t)$ against x, when f is taken as a function of (x-vt) at $t = 0$.

Through the above cases of p = 1 (Kdv) and p = 2 (Mkdv), the equations have shown the simplicity of this method for finding implicit solution.

Now to continue by using this method for the rest value of p i.e. for p > 2, we find that for p = 3 and p > 4 equation (1.2.1) is non integrable and possesses a finite number of conserved quantities [2,3].

3. Thirdly, we will apply this method for p = 4, then the form of the Generalized Kdv equation (1.2.1) is written as

$$u_t + 30u^4 u_x + \gamma u_{3x} = 0. \tag{1.3.33}$$

Using the same set of considerations before, equation (1.3.16) changes to

$$f = \int \left(\frac{D}{H_1}\right)^{-1} \left(-\frac{2}{\gamma}F^6 + \frac{v}{\gamma}F^2 + \frac{2c_1}{\gamma}F + \frac{2c_2}{\gamma}\right)^{-\frac{1}{2}} dF. \tag{1.3.34}$$

To readily get closed-form result, we choose $c_1 = 0$, $c_2 = -2$ and $v = 6$ according to [3] and introducing a new function such that

$$F = (G)^{\frac{1}{2}}. \tag{1.3.35}$$

Now, we make the appropriate choice for $\frac{D}{H_1}$, namely

$$\frac{D}{H_1} = \frac{1}{2(2+G)^{\frac{1}{2}}}. \tag{1.3.36}$$

Upon substituting from (1.3.35) and (1.3.36) into (1.3.34), we get

$$f=\int(2+G)^{\frac{1}{2}}\left(-\frac{v}{3\gamma}G^{3}+\frac{v}{\gamma}G-\frac{2v}{3\gamma}\right)^{\frac{1}{2}}G^{-\frac{1}{2}}\,dG, \tag{1.3.37}$$

or

$$\left(\frac{-v}{3\gamma}\right)^{\frac{1}{2}} f=\int\frac{(2+G)^{\frac{1}{2}}\;G^{-\frac{1}{2}}}{\left(G^{3}-3G+2\right)^{\frac{1}{2}}}\,dG. \tag{1.3.38}$$

Hence, $G^3-3G+2=(G-1)^2\,(G+2)$, we obtain

$$\left(\frac{-v}{3\gamma}\right)^{\frac{1}{2}} f=\int\frac{dG}{G^{\frac{1}{2}}\,(G-1)}\quad. \tag{1.3.39}$$

Let $G=\frac{1}{z^2}$, then

$$\left(\frac{-v}{12\gamma}\right)^{\frac{1}{2}} f=\int\frac{-dz}{\left(1-z^{2}\right)}\quad. \tag{1.3.40}$$

Then, the solution is

$$\left(\frac{-v}{12\gamma}\right)^{\frac{1}{2}} f=\coth^{-1}(z), \tag{1.3.41}$$

and

$$G=\tanh^{2}(\rho)\ , \tag{1.3.42}$$

where $\rho=\left(\frac{-v}{12\gamma}\right)^{\frac{1}{2}} f$, $(v<0\ ,\ \gamma>0)$ or $(v>0\ ,\ \gamma<0)$. Finally, using (1.3.35) we have

$$F(f)=\tanh(\rho)\ , \tag{1.3.43}$$

with $\rho=\left(\frac{-v}{12\gamma}\right)^{\frac{1}{2}} f$, $(v<0\ ,\ \gamma>0)$ or $(v>0\ ,\ \gamma<0)$.

Now, to determine the value of $H_3(f)$, we use (1.3.12), with $H_1(f)=f$, then

$$H_3(f)=(C+1)f - f\int \frac{1}{2\sqrt{(2+F^2)}} df . \tag{1.3.44}$$

Then, we can determine F and $H_3(f)$ as a function in f. In [3] they take $H_1 = f$, v = -1.2, γ = 1 and C = 1 to compute F and $H_3(f)$ against f. Consequently, they calculated $u(x,t)$ and $H_3(x,t)$ against x, when f is taken as a function of (x-vt) at t = 0.

1.4 EXACT SOLUTION OF THE GENERALIZED KDV EQUATION USING THE HYPERBOLIC TANGENT METHOD

1.4 -1 Introduction

In the present section, we will develop and study one of the important and newness a common method, which used to solve the non-linear wave equation. This method is known the hyperbolic tangent (Tanh) method [4]. The presentation a way is simple and quite a powerful tool, for obtaining exact analytical solutions for the PDEs. Also it plays a very important role in physical and chemical sciences.

The main feature of this technique is based on the hypothesis that the travelling wave solution we are looking for may be found and expressed in terms of hyperbolic tangent. This function is then used as a new independent variable. Since all derivatives of a tanh are represented by a tanh itself, and a straightforward analysis can then be carried out, so that the method will be applicable to a large class of nonlinear wave equations.

This technique is originally established by Huibin and Kelin [17], where they used the expansion in terms of a tanh function already to solve a Kdv-Burgers type of equation and then it discussed by Petrsch [33], to treat a schrodinger equation.

Thereafter, the same principle of the (Tanh) method [33] is used in physical science by Malfliet [34], to investigate a set of two nonlinear (dynamical) coupled reaction diffusion equation analytically, where the resulting solutions are corresponding with earlier numerical calculation of the same problem by Merkin and Needham [35].

Quit recently, in the paper of Malfliet [4], the basic of the schematic outline of the (Tanh) method are demonstrated in detail, and it applied for some well-known equations for obtaining travelling wave solutions, such as Burger, Kdv, Mkdv and Fisher equations, where he reported systematized approach depends on the fact that the travelling wave solution must be expressed as a finite power series in y, such that the new independent variable y define as

$$y = \tanh(\xi) = \tanh(c(x - vt)),$$

where the highest power of y is determined by applied the balancing a way. This a way is applied with the Burger's equation, while the two equations Kdv and Mkdv which are discussed in their paper does not handle the balancing operation. The reason for this will become clear in subsection below. The same procedure in [4] is applied on some chemical problems by Ndayirinde and Malfliet [36], where they were find exact solutions of an Isothermal autocatalytic system.

The recently published paper by Malfliet and Hereman I [18] is used a systemized version of the (Tanh) method to solve particular non-linear evolution and wave equations. With the property of the conservative systems the solutions are developed and the technique are improved and systemized through the corporation of boundary condition and the a priori determination of the velocity. So that tedious algebra is avoided. Then, closed form solutions are derived in an elegant and a straightforward way.

In addition, with the aid of the (Tanh) method, Malfliet and Hereman II [37] were able to establish the perturbation procedures, to solve the Kdv-Burger and Mkdv-Burger equations approximately. As a result, a general shock wave profile, with a

perturbative solitary wave contribution superposed, and comparison with the exact solution is made.

Also, the nonlinear evolution equation which describes shock waves in dissipative nonlinear LC Circuit are solved by Malfliet [38] by using the (Tanh) method which illustrated by Malfliet and Hereman I, II [18,37], where the results of exact solution to this problem are the same results which obtained by Watanabe *et al,* where Watanabe *et al* used the reductive perturbation method to solve the same equation.

Then, the ID problem of an incompressible fluid which locally perturbed by a pulse wave of pressure was studied by Malfliet and Ndayirinde, with the aid of the transparent (Tanh) technique, the relevant equation are then solved up to third order.

Our main goal of the following analysis is to determine whether the solitary wave solution of the Generalized Kdv equation can be found with the aid of the (Tanh) technique [4].

1.4-2 Analysis of the Hyperbolic Tangent Method on the Generalized Kdv Equation

In this part, we will apply the principal procedures of the (Tanh) method (Malfliet) [4], for obtaining the exact closed form solution of the Generalized Kdv equation (1.2.1)

For convenience, we want to take full account of the nonlinear property of the problem under study. Since the non-linear waves propagate without change of their shape, we used the behavior of the conservative systems to tackle those kind of problem.

Consider the Generalized Kdv equation (1.2.1), where $u(x,t)$ is a conserved quantity i.e. [$\int_{-\infty}^{\infty} u(x,t)\,dx$ is a constant of motion][18]. In general, to solve nonlinear evolution and wave equations, in the first, we consider that the travelling wave solution u(x,t) (which travel with a constant speed v and vanish as $|x| \to \infty$) introduced in one coordinate

$$\xi = c\,(x - vt), \tag{1.4.1}$$

such that

$$u\,(x\,,t) = \phi\,(\xi), \tag{1.4.2}$$

where the function $\phi(\xi)$ represents the (localized) wave solution, and it has the characteristic that the explicit presence of the positive parameter c (wave number), inversely proportional to the wavelength L, i.e. $L = c^{-1}$. Now, under these assumptions above, equation (1.2.1) (NPDE) reduces to (NODE) as:

$$-cv\,\phi_\xi + c\,(p+1)\,(p+2)\,\phi^p\,\phi_\xi + c^3\,\gamma\phi_{3\xi} \;=\; 0, \tag{1.4.3}$$

where

$$\left.\begin{aligned} \frac{\partial}{\partial t} &\;\to\; -cv\frac{d}{d\xi}, \\ \frac{\partial}{\partial x} &\;\to\; c\,\frac{d}{d\xi}, \\ \frac{\partial^3}{\partial x^3} &\;\to\; c^3\,\frac{d^3}{d\xi^3}. \end{aligned}\right\} \tag{1.4.4}$$

After integrating equation (1.4.3) once w. r. to ξ we obtain

$$-v\,\phi(\xi) + (p+2)\;\phi^{p+1}(\xi) + \gamma\,c^2\,\phi_{2\xi}\,(\xi) \;= C_1\;, \tag{1.4.5}$$

where the associated integration constant C_1 will take to be zero [4]. Next, the crucial step of this method is the introduction of $y = \tanh\,(\xi)$ as a new independent variable, such that the solution we are looking for must be expressed in terms of y. Hence

$$\phi\,(\xi) \;\to\; S\,(y).$$

As a consequence, the corresponding derivatives in (1.4.5) are transformed as follows

$$\frac{d}{d\xi} \to (1 - y^2)\frac{d}{dy}, \tag{1.4.6–a}$$

$$\frac{d^2}{d\xi^2} \rightarrow (1-y^2)\left[-2y\frac{d}{dy} + (1-y^2)\frac{d^2}{dy^2} \right]. \qquad (1.4.6\text{–b})$$

Substituting from (1.4.6-b) in (1.4.5), equation (1.4.5) can be written in terms of S (y) as

$$-v\,S(y) + (p+2)\,S^{p+1}(y) + c^2\gamma\,(1-y^2)\left[-2y\frac{d\,S(y)}{dy} + (1-y^2)\frac{d^2\,S(y)}{dy^2} \right] = 0. \quad (1.4.7)$$

Then, the solution S (y) will expressed as a finite power series in y. Such expansions, were already used by Malfliet [4] as

$$S(y) = \sum_{n=0}^{N} a_n\, y^n. \qquad (1.4.8)$$

Now, the highest power of y (i.e. N) will be determined by the balancing between the highest order of the linear term (essentially the highest derivative) and the highest order of non-linear term. When we apply this operation on the equation (1.4.7), the parameter N can be found as

$$(p+1)N = 3 + N - 1 \text{ or } 4 + N - 2,$$

so that

$$N = \frac{2}{p}. \text{ (p is a positive integer)} \qquad (1.4.9)$$

Notice that, equation (1.4.9) gives us fractional number, if the parameter p take value larger than 2. Consequently, we are not able to apply the balancing operation for all value of p. So, we will discuss two important particular cases of the Generalized Kdv equation (1.2.1).

In the first case, for p = 1 (Kdv), equation (1.2.1) transforms to (1.2.2). If we applying the same rules as before, equation (1.4.7) changes to

$$-v\,S(y) + 3\,S^2(y) + \gamma c^2\,(1-y^2)\left[-2y\frac{d\,S(y)}{dy} + (1-y^2)\frac{d^2\,S(y)}{dy^2} \right] = 0, \quad (1.4.10)$$

such that the solution S(y) is written as equation (1.4.8), where the parameter N is determine by using the equation (1.4.9). However, form the fact that N = 2, we have able to introduce S (y) as

$$S(y) = a_0 + a_1 y + a_2 y^2 \quad . \tag{1.4.11}$$

Upon substituting (1.4.11) into (1.4.10), we get

$$\begin{aligned} &- v\,(a_0 + a_1 y + a_2 y^2) + 3(a_0 + a_1 y + a_2 y^2)^2 + \\ &c^2\gamma\,(1 - y^2)\left[-2y\,(a_1 + 2a_2 y) + (1 - y^2)\,(2a_2)\right] = 0. \end{aligned} \tag{1.4.12}$$

As a result to collecting all terms with the same power in y^m (m = 0, 1, 2, 3, 4), which must vanish to give

$$\left.\begin{aligned} &y^0 \text{ coeff.}: \; -va_0 + 3a_0^2 + 2c^2\gamma a_2 = 0, \\ &y^1 \text{ coeff.}: \; -va_1 + 6a_0a_1 - 2\gamma c^2 \; a_1 = 0, \\ &y^2 \text{ coeff.}: \; -va_2 + 3a_1^2 + 6a_0 a_2 - 8\gamma c^2 a_2 = 0, \\ &y^3 \text{ coeff.}: \; 6a_1a_2 + 2\gamma c^2 a_1 = 0, \\ &y^4 \text{ coeff.}: \; 3a_2^2 + 6c^2\gamma a_2 = 0. \end{aligned}\right\} \tag{1.4.13}$$

After some algebra, the unknowns are easily found (choose c and γ, as free parameters).

Case (1) the suitable variables are

$$a_0 = 2c^2\gamma, \quad a_1 = 0, \quad a_2 = -2c^2\gamma, \quad v = 4c^2\gamma. \tag{1.4.14}$$

Case (2) the suitable variables are

$$a_0 = \frac{2c^2\gamma}{3}, \quad a_1 = 0, \quad a_2 = -2c^2\gamma, \quad v = -4c^2\gamma. \tag{1.4.15}$$

Consequently, there are two solutions corresponding to the previous two cases (1.4.14) and (1.4.15) respectively as

$$u(x,t) = 2c^2\gamma \operatorname{sech}^2\left(c\,(x - 4c^2\gamma t)\right), \tag{1.4.16}$$

and

$$u(x,t) = 2c^2\gamma(\mathrm{sech}^2(c(x+4c^2\gamma t)) - \frac{2}{3}). \qquad (1.4.17)$$

Notice that, the first solution (1.4.16) is agreeing with the same solution that produced by Hereman *et al* [6], and the second solution (1.4.17) also represents a solution with a factor translation $-\frac{4}{3}c^2\gamma$. By another technique, which is employed by Malfliet [4], we observe from the structure of equation (1.4.10) that S (y) should be proportional to $(1-y^2)$. So we are able to introduce S (y) as

$$S(y) = \beta(1-y^2). \qquad (1.4.18)$$

Substitution into (1.4.10) and then dividing by $\beta\ (1-y^2)$, we obtain

$$-v+3\beta(1-y^2)+c^2\gamma\left(4y^2-2(1-y^2)\right)=0. \qquad (1.4.19)$$

After some algebra we arrive at a power series in y up to second order. Hence the following relations arise:

$$\left.\begin{array}{l} y^0 \text{ coeff.}: -v+3\beta-2c^2\gamma=0, \\ y^2 \text{ coeff.}: -3\beta+6c^2\gamma=0, \end{array}\right\} \qquad (1.4.20)$$

we choose c and γ as free parameters, then the other variables are found. i.e.

$$\beta = 2c^2\gamma \quad \text{and} \quad v = 4c^2\gamma.$$

Finally, the general solution of (1.2.2) is agreeing with the first solution (1.4.16), which it appeared when we introduced the solution S (y) in (1.4.11).

In the second case, when $p=2$, equation (1.2.1) changes to (1.2.3). Now, by using the same asymptotic procedure above, equation (1.4.7) can be expressed as

$$-vS(y)+4S^3(y)+\gamma c^2(1-y^2)\left[-2y\frac{dS(y)}{dy}+(1-y^2)\frac{d^2S(y)}{dy^2}\right]=0. \qquad (1.4.21)$$

The solution S(y) is a power series in y in which the highest possible power, found through the balancing procedure, equals 1. So that

$$S(y)=\sum_{n=0}^{1}a_n y^n = a_0 + a_1 y \qquad (1.4.22)$$

Substituting (1.4.22) into (1.4.21), we obtain

$$-va_0 + 4a_0^3 + [-va_1 - 2c^2\gamma a_1 + 12a_0^2 a_1]y + 12a_0 a_1^2 y^2 + [4a_1^3 + 2\gamma c^2 a_1]y^3 = 0. \quad (1.4.23)$$

Then the coefficients of different power of y will vanish. i.e.

$$\left.\begin{aligned} &y^0\text{coeff.}: -va_0 + 4a_0^3 = 0, \\ &y^1\text{coeff.}: -va_1 + 12a_0^2 a_1 - 2\gamma c^2 a_1 = 0, \\ &y^2\text{ coeff.}: 12a_0 a_1^2 = 0, \\ &y^3\text{ coeff.}: 4a_1^3 + 2\gamma c^2 a_1 = 0. \end{aligned}\right\} \quad (1.4.24)$$

Solving the previous set of equations (choose c and γ, as free parameters), we obtain the following suitable results

$$a_0 = 0, \qquad a_1^2 = -\frac{\gamma c^2}{2}, \qquad v = -2\gamma c^2.$$

Then yields the solution

$$u(x,t) = \pm c\sqrt{\frac{-\gamma}{2}}\tanh(c(x + 2\gamma c^2 t)), \quad (1.4.25)$$

which is represented a shock wave as a well-known solution of the Mkdv equation for negative γ. (this solution is not appearing in [4]).

Now, we will try to define S(y) such that the other choice solution originates from the structure of equation (1.4.21), which suggests the ansatz [4]

$$S(y) = \beta(1-y^2)^{\frac{1}{2}}. \quad (1\ 4.26)$$

After substitution and dividing by $\beta(1-y^2)^{\frac{1}{2}}$, equation (1.4.21) transforms to

$$-v + 4\beta^2(1-y^2) + \left[2c^2\gamma y^2 - c^2\gamma\left(1-y^2\right) - c^2\gamma y^2\right] = 0. \quad (1.4.27)$$

Only terms proportional to y^2 and y^0 are left. Thus

$$\left.\begin{aligned} &y^0\text{ coeff.}: -v + 4\beta^2 - c^2\gamma = 0, \\ &y^2\text{ coeff.}: -4\beta^2 + 2c^2\gamma = 0. \end{aligned}\right\} \quad (1.4.28)$$

Let c and γ be free parameters, then

$$\beta = c\sqrt{\frac{\gamma}{2}}, \qquad v = c^2 \gamma.$$

Consequently, the exact solution of (1.2.3) takes the following form

$$u(x,t) = c\sqrt{\frac{\gamma}{2}} \operatorname{sech}(c(x - c^2 \gamma t)). \qquad (1.4.29)$$

Now, from the two cases above, we are able to apply the same procedures of the (Tanh) method to find the exact solution of the Generalized Kdv equation (1.2.1). For the general case (1.4.7), we can define the solution S(y) as:

$$S(y) = \beta(1 - y^2)^{\frac{1}{p}}, \qquad (1.4.30)$$

with $y = \tanh(\xi)$. Upon substituting into (1.4.7) and then dividing over the scaling $\beta(1-y^2)^{\frac{1}{p}}$, we readily arrive

$$-v + (p+2)\beta^p (1-y^2) + \frac{4}{p}\gamma c^2 y^2 - \frac{2c^2\gamma}{p}(1-y^2) + \frac{4\gamma c^2 y^2 (1-p)}{p^2} = 0. \quad (1.4.31)$$

Next, collecting all terms with the same power in $y^m (m = 0,2)$, which must vanish, we get

$$\left.\begin{aligned} &y^0 \text{coeff.}: \ -v + (p+2)\beta^p - \frac{2c^2\gamma}{p} = 0, \\ &y^2 \text{ coeff.}: -(p+2)\beta^p + \frac{2\gamma c^2}{p} + \frac{4c^2\gamma}{p^2} = 0. \end{aligned}\right\} \qquad (1.4.32)$$

The unknowns are now easily found, through applying simple algebra to be

$$\beta^p = \frac{2c^2\gamma}{p^2}, \qquad v = \frac{4c^2\gamma}{p^2}. \qquad (1.4.33)$$

If we choose c and γ as free parameters, the general solution of (1.4.30) is written as

$$S(y) = \left(\frac{2c^2\gamma}{p^2}(1-y^2)\right)^{\frac{1}{p}}. \qquad (1.4.34)$$

Hence, we obtain the well-known solitary wave solution of (1.2.1) as

$$u(x,t)=\left(\frac{2c^2\gamma}{p^2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}\left(c\left(x-\frac{4c^2\gamma}{p^2}t\right)\right). \qquad (1.4.35)$$

The solitary wave solution (1.4.35) is similar to the result solution, which will it obtain in section (2.3) when $c=\frac{p}{2}\sqrt{\frac{v}{\gamma}}$, if we choose v as a free parameter.

Chapter(2)

CHAPTER (2)

THE SERIES EXPANSION METHOD

2.1 INTRODUCTION

In this chapter, we are concerned with one of the most important and well-known analytical method for solving various nonlinear partial differential equations (NPDEs). Although, it is appears with different names such as a series expansion method, direct algebraic method and real exponential approach, it has the same form and technique. The method is classified as a direct method.

This technique (real exponential approach) is originally established by Korpel [16] to demonstrate an alternative, synthesis for using the real exponential wave characteristics of the linear medium rather than the complex. The application example of his treatment is the Kdv equation, as a model of soliton formation and the nonlinearity mediates the coupling between real exponential waves.

In the same way, Hermann *et al* [5,6] introduced a physical approach to derive an explicit solitary wave solution of nonlinear evolution and wave equations. They also used the same technique with more mathematical rigorous and systematic procedure to include the derivation of solitary wave solutions with various nonlinear partial differential equations. The physical interpretation is still given by mixing the elementary solution of the linear part in the various (NPDE) due to the nonlinear part. The first and the second equations (namely, the Kdv and Mkdv equations) in the class of the Generalized Kdv equation are treated among the studied equations in [6].

By the same method (the series expansions analysis), Coffey [26] studied two Kdv like equations, one with fifth-degree non-linearity and the other is combined Kdv and Mkdv equations. The obtained closed form solution of them is agreeing with Dey [27].

The application example of this chapter is the Generalized Kdv equation [1,2], using the series expansion method, to obtain a closed form solution (solitary wave).

Also, some new forms of solution for Kdv and Mkdv equations are obtained [6] by using the same technique.

The organization of this chapter is as follows:

1) In section (2.2), the solution method for a general partial differential equation is outlined in some stepping to introduce the series solution, formal of obtained recursion relation (RR) and how to solve it.

2) In section (2.3), we apply the series expansion technique on the Generalized Kdv with $(p+1)$ non-linearity terms where p is almost a positive integer. The constant term (c_1) in the series and the constant of integration are chosen equal to zero with the general parameter p. The obtained recursion relation (RR) in this case is not easily solved with dummy p. Consequently, we treat the first cases of $p = 1, 2, 3, 4, 5, 6$ for determining the general coefficients of the polynomial $\{a_n\}_{n=1}^{\infty}$ due to solving the (RR) in each separating case. That is able us to deduce it with general p and the solitary wave solutions (kink and antikink) are obtained.

3) Section (2.4) is devoted to study the Kdv and Mkdv equations (i.e. p = 1 and 2 in equation (1.2.1)), using the series expansion technique. The solution is obtained, when c_1 takes different values than zero and the constant of integration is still of zero value.

4) In the last section (2.5) of this chapter, with the same technique, we deduce the solitary wave solution of the Kdv equation (1.2.2) corresponding to $c_1 \neq 0$ and the integration constant, which takes different values.

The results of the solitary wave solutions for each separation case are represented in three dimensions plot (using Mathematica 3.0 program) [9]. The comparison of the obtained solutions with the corresponding one given in [6] is illustrated.

2.2 DESCRIPTION OF THE METHOD

The method is summarized in the following steps as a general application on a general non-linear form of equations.

1) First, we convert the nonlinear partial differential equation with x and t as the space and time coordinates respectively, to a corresponding ordinary one. That is by introducing a travelling frame of reference $\xi = x - vt$ such that a travelling wave solution

$$u(x,t) = \phi(\xi),$$

where ξ is the new independent variable, and v is the constant anticipated velocity of the solitary wave.

2) Integrating the resulting ODE as many times as necessary, e.g. for those, which contain a first derivative of t we need to integrate once only, while the wave equation needs twice integration.

3) If we restrict ourselves with the solitary wave solution, it possibly adds a constant term c_1 in its real expansion, i.e. we introduce $\phi = c_1 + \hat{\phi}$ in the nonlinear partial differential equation . The constant term c_1 is generally related to the singular points of the vector field in phase space as in Coffey [26]. Some times, it is taken equal to zero as indicating in the following treatments (as in section (2.3)).

4) If we consider the linear part of the final equation of $\hat{\phi}$ i.e. neglecting the nonlinear terms or taking the coefficients of them equal to zero, we can put the solution in the form $\exp[\pm K(v)\ \xi]$ where K (v) is a real function. Consequently, it is required that the constant term in this equation equal to zero or we may impose certain conditions on c_1 as well as the constant of integration.

5) For the sake of mathematical convenience, we may need to normalize few coefficients of the nonlinear terms by choosing a simple scaling transformation. e.g.

$$\hat{\phi} = \eta\tilde{\phi}.$$

6) After the scaling, the solution $\tilde{\phi}$ of the non-linear equation can be expressed as

$$\tilde{\phi}(\xi) = \sum_{n=1}^{\infty} a_n g^n(\xi), \qquad \text{where} \quad g(\xi) = \exp(\pm K\xi).$$

Then the problem is converted to determine the coefficients a_n and the general form of them. This is beginning by writing the recursion relation (RR) and solve it.

7) The nonlinear equation on $\tilde{\phi}$, generally, contains non-linear and dispersion and/or dissipation terms of arbitrary order and degree. Consequently, the substitution by $\tilde{\phi}$ in the general nonlinear equation and using, the extension of Cauchy's product rule (I), (II) as in Gradshtyn and Ryzhik [39]

$$\left(\sum_{n=1}^{\infty} a_n g^n\right)^I = \sum_{n=I}^{\infty} \sum_{r=I-1}^{n-1} \cdots\cdots\cdots \sum_{m=2}^{k-1} \sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} \cdots\cdots\cdots a_{n-r} g^n, \quad \text{(I)}$$

$$\tilde{\phi}_{q\xi} = (-K)^q \sum_{n=1}^{\infty} n^q a_n g^n, \quad \text{(II)}$$

such that, the powers of (-K) are absorbed in the constant C_H, ..., C_L (if they are not eliminated in the scaling), then the general nonlinear equation leads to

$$\sum_{n=1}^{\infty} p(n) a_n g^n + C_H \sum_{n=H}^{\infty} \sum_{r=H-1}^{n-1} \cdots\cdots \sum_{m=2}^{k-1} \sum_{\ell=1}^{m-1} (\ell)^{\mu_1} (m-\ell)^{\mu_2} \cdots\cdots$$
$$(n-r)^{\mu_H} a_\ell a_{m-\ell} \cdots\cdots\cdots a_{n-r} g^n + \cdots\cdots\cdots + C_L \sum_{n=L}^{\infty} \sum_{r=L-1}^{n-1} \cdots\cdots$$
$$\sum_{m=2}^{k-1} \sum_{\ell=1}^{m-1} (\ell)^{\nu_1} (m-\ell)^{\nu_2} \cdots\cdots\cdots (n-r)^{\nu_L} a_\ell a_{m-\ell} \cdots\cdots\cdots a_{n-r} g^n = 0,$$

where $\sum_{n=1}^{\infty} p(n) a_n g^n$ represents the linear terms, and the degree of polynomial $p(n)$ is equal to the highest order of dispersion (or dissipation) minus the number of integration carried out. Also, H and L refer to the highest and lowest order of the non-linearity $\mu_1, \nu_1, \ldots\ldots\ldots \mu_H, \nu_L$. indicates the order of the derivative associated with $\tilde{\phi}$ in the nonlinear term.

8) Although, the main equation in step (7) contains a summation in the first terms starting from n =1, the other summations at least begin with $L \geq 2$ in the non-linear terms. Consequently, if $p(j) = 0$, $j = 1,2,3\ldots L-1$, at least one of the coefficients $a_1, a_2, \ldots, a_j \ldots a_{L-1}$ is arbitrary. On the other hand, when $p(j) \neq 0$ for the same j, then at least one of these coefficients will be zero. This is motivated to delete the summation over n, for $n \geq H$, to obtain the recursion relation (RR) in the following form

$$p(n)a_n + C_H \sum_{r=H-1}^{n-1} \ldots \sum_{m=2}^{k-1} \sum_{\ell=1}^{m-1} (\ell)^{\mu_1} (m-\ell)^{\mu_2} \ldots (n-r)^{\mu_H} a_\ell a_{m-\ell} \ldots a_{n-r}$$
$$+ \ldots + C_L \sum_{r=L-1}^{n-1} \ldots \sum_{m=2}^{k-1} \sum_{\ell=1}^{m-1} (\ell)^{\nu_1} (m-\ell)^{\nu_2} \ldots (n-r)^{\nu_L} a_\ell a_{m-\ell} \ldots a_{n-r} = 0,$$
$$n \geq H.$$

9) When we obtain the general form of a_n, back substitution is required to get the solution of the original equation.

The previous technique is illustrated in the following sections to obtain a general solitary wave solution of the Generalized Kdv equation especially Kdv and Mkdv equations.

2.3 SOLUTION OF THE GENERALIZED KORTEWEG DE VRIES EQUATION USING THE SERIES EXPANSION METHOD

In this section, we apply the series expansion method (explained generally in the previous section) on the Generalized Kdv equation [1,2], the Generalized Kdv equation is written in the well-known form (as in section (1.2)). i.e.

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{3x} = 0, \tag{2.3.1}$$

where γ is an arbitrary real number and p is almost a positive integer. The subscripts in (2.3.1) of u_t and u_x refer to the partial derivatives of u (x, t) with respect to time (t) and space variable (x) respectively. To apply the procedures of the series expansion method, we introduce

$$u(x,t) = \phi(\xi), \qquad \xi = x - vt, \tag{2.3.2}$$

such that ξ and v are still defined as in section (2.2). This assumption transforms the (NPDE) (2.3 1) to the following (NODE)

$$-v\phi_\xi + (p+1)(p+2)\phi^p\phi_\xi + \gamma\phi_{3\xi} = 0, \tag{2.3.3}$$

where

$$\frac{\partial}{\partial x} = \frac{d}{d\xi}\frac{d\xi}{dx} = \frac{d}{d\xi}, \quad \frac{\partial}{\partial t} = \frac{d}{d\xi}\frac{d\xi}{dt} = -v\frac{d}{d\xi},$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{d}{d\xi}\right) = \frac{d^2}{d\xi^2},$$

$$\frac{\partial^3}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{d^2}{d\xi^2}\right) = \frac{d^3}{d\xi^3}.$$

Now, integration (2.3.3) to obtain

$$-v\phi + (p+2)\phi^{(p+1)} + \gamma\phi_{2\xi} = h, \tag{2.3.4}$$

where h is the integration constant. It may be taken equal to zero (local solitary wave) or some other values of certain cases of the Kdv and Mkdv equations [6] which deducing in the following sections.

As we mentioned, it is useful to put a constant term in the real exponential technique for a solitary wave solution [6]. Consequently, the solution can be written as

$$\phi = c_1 + \hat{\phi}. \qquad (2.3.5)$$

Substituting (2.3.5) into (2.3.4), we have

$$-v\left(c_1+\hat{\phi}\right)+(p+2)\left(c_1+\hat{\phi}\right)^{p+1}+\gamma\left(c_1+\hat{\phi}\right)_{2\xi}=h, \qquad (2.3.6\text{-a})$$

or

$$\left[-v\,c_1+(p+2)c_1^{p+1}-h\right]+\left[-v+(p+1)(p+2)c_1^{p}\right]\hat{\phi}+$$
$$(p+2)\left[\frac{(p+1)p}{2!}c_1^{p-1}\hat{\phi}^2+\frac{(p+1)p(p-1)}{3!}c_1^{p-2}\hat{\phi}^3+\dots\dots\dots\right]+\gamma\,\hat{\phi}_{2\xi}=0. \qquad (2.3.6\text{-b})$$

The linear part of equation (2.3.6-b) has a solution of the form exp ($\pm K\xi$) if the constant term equal to zero. i .e.

$$-v\,c_1+(p+2)c_1^{p+1}-h=0, \qquad (2.3.7)$$

and the wave number is given by

$$K^2=\frac{v-(p+1)(p+2)\,c_1^{p}}{\gamma}, \qquad (2.3.8)$$

then $v>(p+1)(p+2)c_1^p$ at positive γ and $v<(p+1)(p+2)c_1^p$ when γ is negative for real K.

Now, we choose (the constant of integration) h equal to zero, then (2.3.7) and (2.3.8) give

$$c_1=0, \qquad \text{and } K^2=\frac{v}{\gamma}, \qquad (2.3.9\text{-a})$$

$$c_1^p=\frac{v}{p+2}, \qquad \text{and } K^2=\frac{-pv}{\gamma}. \qquad (2.3.9\text{-b})$$

The second case (2.3.9-b) will be relegated to the following sections. Then, upon considering the first one (2.3.9-a), equation (2.3.6) becomes the same cases as (2.3.4) with $h=0$ and $\hat{\phi}$ instead of ϕ, namely:

$$-v\hat{\phi}+(p+2)\hat{\phi}^{(p+1)}+\gamma\hat{\phi}_{2\xi}=0. \qquad (2.3.10)$$

We normalize the nonlinear equation (2.3.10) by using a scaling transform $\hat{\phi} = \eta\tilde{\phi}$, and then it gives

$$-v\tilde{\phi} + (p+2)\eta^P\tilde{\phi}^{P+1} + \gamma\tilde{\phi}_{2\xi} = 0, \tag{2.3.11}$$

choosing

$$\eta^P = \frac{v}{P+2}, \tag{2.3.12}$$

we have

$$-v\tilde{\phi} + v\tilde{\phi}^{(p+1)} + \gamma\tilde{\phi}_{2\xi} = 0. \tag{2.3.13}$$

Let us now, define

$$\tilde{\phi}(\xi) = \sum_{n=1}^{\infty} a_n g^n(\xi), \tag{2.3.14}$$

where $g_1 = \frac{1}{g_2} = g = e^{-K\xi}$ as a result from $K_1 = -K_2 = K = \left(\frac{v}{\gamma}\right)^{\frac{1}{2}}$.

Substituting from (2.3.14) into (2.3.13), we obtain

$$-v\sum_{n=1}^{\infty} a_n g^n + v\left(\sum_{n=1}^{\infty} a_n g^n\right)^{p+1} + \gamma\left(\sum_{n=1}^{\infty} a_n g^n\right)_{2\xi} = 0. \tag{2.3.15}$$

Using Cauchy's rule [39] in the intermediate term (non-linear part) and the decaying from of g (ξ) as above, such that $\gamma K^2 = v$ (2.3.9-a), we have

$$\sum_{n=1}^{\infty}(n^2-1)a_n g^n + \sum_{n=p+1}^{\infty}\sum_{n_1=p}^{n-1} \cdots\cdots \sum_{n_p=1}^{n_{p-1}-1} a_{n-n_1} a_{n_1-n_2} \cdots\cdots a_{n_{p-1}-n_p} a_{n_p} g^n = 0, \tag{2.3.16}$$

it is required that a_1 is generally arbitrary and $a_2 = a_3 = a_4 = \cdots\cdots = a_p = 0$, because the second summation begins at $n = p+1$. If we omit the summation over n as mentioned in step (8) of the previous section, the required recursion relation (RR) for $n \geq p+1$ is obtained in the following form

$$(n^2-1)a_n + \sum_{n_1=p}^{n-1}\sum_{n_2=p-1}^{n_1-1} \cdots\cdots \sum_{n_p=1}^{n_{p-1}-1} a_{n-n_1} a_{n_1-n_2} \cdots a_{n_{p-1}-n_p} a_{n_p} = 0. \tag{2.3.17}$$

Now, we need to determine the coefficient a_n and its general form (namely, the solution of (RR) (2.3.17)). But this is not easy or possible with dummy p and it motivates as to consider some different choices of p as follows:

1. Starting with $p = 1$, i.e. we study the Kdv equation.

$$u_t + 6uu_x + \gamma u_{3x} = 0. \tag{2.3.18}$$

By performing the same analysis and suitable transformations as above, the resulting (RR) is

$$\left(n^2 - 1\right)a_n + \sum_{n_1=1}^{n-1} a_{n_1} a_{n-n_1} = 0, \quad n \geq 2, \tag{2.3.19}$$

where a_1 is arbitrary coefficient. The (RR) (2.3.19) is readily solved and the first few coefficients which depending on a_1 are calculating to give

$$a_2 = \frac{-2a_1^2}{6},\ a_3 = \frac{3a_1^3}{6^2},\ a_4 = \frac{-4a_1^4}{6^3},\ a_5 = \frac{5a_1^5}{6^4},\ a_6 = \frac{-6a_1^6}{6^5},\ldots\ldots$$

The general form of these coefficients can be easily deduced, then

$$a_{n+1} = \frac{(-1)^n (n+1) a_1^{n+1}}{6^n}, \qquad n = 0,1,2,\ldots. \tag{2.3.20}$$

2. If we put $p = 2$ in equation (2.3.1), we get the Modified kdv (Mkdv) equation of the form

$$u_t + 12u^2 u_x + \gamma u_{3x} = 0. \tag{2.3.21}$$

Repeating the same technique as mentioned before, we can get the following

$$\left(n^2 - 1\right)a_n + \sum_{n_1=2}^{n-1} \sum_{n_2=1}^{n_1-1} a_{n-n_1} a_{n_1-n_2} a_{n_2} = 0, \quad n \geq 3. \tag{2.3.22}$$

Upon, finding out the first few coefficients such that a_1 is arbitrary and $a_2 = 0$, the solution of the (RR) (2.3.22) leads to

$$a_3 = \frac{-a_1^3}{2^3}, \qquad a_4 = 0, \qquad a_5 = \frac{a_1^5}{2^6}, \qquad a_6 = 0,$$
$$a_7 = \frac{-a_1^7}{2^9}, \qquad a_8 = 0, \qquad a_9 = \frac{a_1^9}{2^{12}}, \qquad a_{10} = 0, \ldots..$$

It is easy to write down the general term of this polynomial as

$$a_{2s+i} = \begin{Bmatrix} \dfrac{(-1)^s a_1^{2s+1}}{2^{3s}} & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1 \end{Bmatrix}, \tag{2.3.23}$$

where i = 1,2 for every s= 0,1,2,3...

3. We have used the series expansion method to find the (RR) corresponding to p = 3 and solve it. Consequently, equation (2.3.1) becomes

$$u_t + 20u^3 u_x + \gamma u_{3x} = 0. \tag{2.3.24}$$

Applying the general technique, we can get the following (RR)

$$(n^2 - 1)a_n + \sum_{n_1=3}^{n-1} \sum_{n_2=2}^{n_1-1} \sum_{n_3=1}^{n_2-1} a_{n-n_1} a_{n_1-n_2} a_{n_2-n_3} a_{n_3} = 0, \qquad n \geq 4. \tag{2.3.25}$$

In this case, we deduce that a_1 is still arbitrary and $a_2 = a_3 = 0$. Consequently, the first few coefficients are given by

$$a_4 = -\frac{a_1^4}{15}, \quad a_5 = a_6 = 0, \quad a_7 = \frac{a_1^7}{15 \times 12}, \quad a_8 = a_9 = 0,$$

$$a_{10} = -\frac{a_1^{10}}{(15)^2 \times 9}, \quad a_{11} = a_{12} = 0, \quad a_{13} = \frac{11\, a_1^{13}}{(15)^3 \times 72}, \quad a_{14} = a_{15} = 0,$$

$$a_{16} = -\frac{77\, a_1^{16}}{(15)^5 \times 24}, \quad a_{17} = a_{18} = 0, \quad a_{19} = \frac{7 \times 11 \times 17\, a_1^{19}}{(15)^6 \times 12 \times 24}, \ldots\ldots$$

then the general coefficient of this polynomial is

$$a_{3s+i} = \begin{cases} \dfrac{(-1)^s a_1^{3s+1} (3s-1)!_3}{3^s.(10)^s.\, s!}, & \text{if } i = 1 \\ 0 & \text{if not} \end{cases} \qquad (2.3.26)$$

It holds when i = 1,2,3 for every s = 0,1,2,3... where (3s - 1)!$_3$ = (3s-1)!!! = (3s-1) (3s-4)...8×5×2, and we define $(-1)!_3 = 1$ for s=0.

4. For p= 4, equation (2.3.1) of our consideration becomes

$$u_t + 30u^4 u_x + \gamma\, u_{3x} = 0. \qquad (2.3.27)$$

By applying the formal steps of the method in this case, we have (RR) in the form:

$$(n^2 - 1)a_n + \sum_{n_1=4}^{n-1} \sum_{n_2=3}^{n_1-1} \sum_{n_3=2}^{n_2-1} \sum_{n_4=1}^{n_3-1} a_{n-n_1} a_{n_1-n_2} a_{n_2-n_3} a_{n_3-n_4} a_{n_4} = 0, \qquad (2.3.28)$$
$$n \geq 5.$$

The (RR) (2.3.28) is solved, such that a_1 is arbitrary and $a_2 = a_3 = a_4 = 0$ to determine the rest coefficients as:

$$a_5 = -\frac{a_1^5}{24}, \quad a_6 = a_7 = a_8 = 0, \quad a_9 = \frac{3a_1^9}{(24)^2 \times 2}, \quad a_{10} = a_{11} = a_{12} = 0,$$
$$a_{13} = -\frac{5a_1^{13}}{(24)^3 \times 2}, \; a_{14} = a_{15} = a_{16} = 0, \; a_{17} = \frac{35a_1^{17}}{(24)^4 \times 8}, \; a_{18} = a_{19} = a_{20} = 0,$$
$$a_{21} = \frac{-63a_1^{21}}{(24)^5 \times 8}, \ldots\ldots\ldots$$

and the general term of this polynomial can be easily found in the form

$$a_{4s+i} = \begin{cases} \dfrac{(-1)^s a_1^{4s+1} (2s-1)!_2}{(24)^s .s!} & \text{if } i = 1 \\ 0 & \text{if not} \end{cases}, \qquad (2.3.29)$$

where i = 1,2,3,4 for every s = 0,1,2,3,...and (2s − 1)!$_2$ = (2s-1)!!= (2s-1)(2s-3) ...×5×3×1, if we define $(-1)!_2 = 1$ when s = 0.

5. In this case, we consider equation (2.3.1) at p = 5, then

$$u_t + 42\,u^5 u_x + \gamma u_{3x} = 0\,. \tag{2.3.30}$$

As mentioned above, we are able to express the (RR) (after some analysis of the used technique) as

$$\left(n^2-1\right)a_n + \sum_{n_1=5}^{n-1}\sum_{n_2=4}^{n_1-1}\sum_{n_3=3}^{n_2-1}\sum_{n_4=2}^{n_3-1}\sum_{n_5=1}^{n_4-1} a_{n-n_1}a_{n_1-n_2}a_{n_2-n_3}a_{n_3-n_4}a_{n_4-n_5}a_{n_5} = 0, \quad n \ge 6, \tag{2.3.31}$$

we find that a_1 is arbitrary and $a_2 = a_3 = a_4 = a_5 = 0$. Consequently, the remainder coefficients are

$$a_6 = \frac{-2a_1^6}{5\times(14)}, \qquad a_7 = a_8 = a_9 = a_{10} = 0,$$
$$a_{11} = \frac{2\times 7a_1^{11}}{5^2\times(14)^2\times 2}, \qquad a_{12} = a_{13} = a_{14} = a_{15} = 0,$$
$$a_{16} = \frac{-2\times 7\times 12\; a_1^{16}}{5^3\times(14)^3\times 6}, \qquad a_{17} = a_{18} = a_{19} = a_{20} = 0\;,$$
$$a_{21} = \frac{2\times 7\times 12\times 17\; a_1^{21}}{5^4\times(14)^4\times 24}, \qquad a_{22} = a_{23} = a_{24} = a_{25} = 0\;,$$
$$a_{26} = \frac{-2\times 7\times 12\times 17\times 22\; a_1^{26}}{5^5\times(14)^5\times 120}, \ldots\ldots$$

and it is easy to write down the general form of them. i.e.

$$a_{5s+i} = \begin{Bmatrix} \dfrac{(-1)^s a_1^{5s+1}(5s-3)!_5}{(14)^s.5^s s!} & \text{if} \quad i=1 \\ 0 & \text{if} \quad \text{not} \end{Bmatrix}, \tag{2.3.32}$$

where i = 1,2,3,4,5 for every s = 0,1,2,3,...and $(5s\text{-}3)!_5 = (5s\text{-}3)!!!!! = (5s\text{-}3)(5s\text{-}8)\ \ldots\ 17\times12\times7\times2$, if we define $(-3)!_5 = 1$ for s=0.

6. Let p = 6, the natural proposal is to determine the (RR) and solve it. Then, equation (2.3.1) takes the form

$$u_t + 56u^6 u_x + \gamma u_{3x} = 0. \tag{2.3.33}$$

Then the principle technique yields the following (RR)

$$(n^2 - 1)a_n + \sum_{n_1=6}^{n-1} \sum_{n_2=5}^{n_1-1} \sum_{n_3=4}^{n_2-1} \sum_{n_4=3}^{n_3-1} \sum_{n_5=2}^{n_4-1} \sum_{n_6=1}^{n_5-1} a_{n-n_1} a_{n_1-n_2} a_{n_2-n_3} a_{n_3-n_4} a_{n_4-n_5} a_{n_5-n_6} a_{n_6} = 0, \quad n \ge 7, \tag{2.3.34}$$

where a_1 is arbitrary and $a_2 = a_3 = a_4 = a_5 = a_6 = 0$. On solving the (RR) (2.3.34), we can obtain the first few coefficients as:

$$a_7 = \frac{-a_1^7}{48}, \quad a_8 = a_9 = a_{10} = a_{11} = a_{12} = 0,$$

$$a_{13} = \frac{2a_1^{13}}{(48)^2}, \quad a_{14} = a_{15} = a_{16} = a_{17} = a_{18} = 0,$$

$$a_{19} = \frac{-7\times 2}{3\times(48)^3} a_1^{19}, \quad a_{20} = a_{21} = a_{22} = a_{23} = a_{24} = 0,$$

$$a_{25} = \frac{5\times 7}{3\times(48)^4} a_1^{25}, \quad a_{26} = a_{27} = a_{28} = a_{29} = a_{30} = 0, \ldots..$$

and generally, the form of a_n is

$$a_{6s+i} = \begin{Bmatrix} \dfrac{(-1)^s a_1^{6s+1} (3s-2)!_3}{(24)^s . 2^s s!} & \text{if } i = 1 \\ 0 & \text{if not} \end{Bmatrix}, \tag{2.3.35}$$

where i = 1, 2, 3, 4, 5, 6 for every s = 0,1,2,3... and (3s-2)!$_3$ = (3s-2)!!! = (3s-2) (3s-5) . . . 10×7×4×1, as special for s = 0 we define (-2)!$_3$ = 1.

A computer program is written to calculate the coefficients of the polynomial $\{a_n\}_{n=1}^{N}$ (at finite N) for each separation case of p = 1,2,3,4,5,6 (using Mathematica 3.0 program) [9] (see Appendix (B)) and the results are the same as reported above.

It may be continued to deduce another forms of the general coefficient for higher p as required. But these general forms of a_n in (2.3.20), (2.3.23), (2.3.26), (2.3.29), (2.3.32) and (2.3.35), corresponding to p = 1,2,3,4,5,6 are as many as we need. Then, the general coefficient a_n due to solving the (RR) (2.3.17) with general p can be written in the following form

$$a_{ps+i} = \left\{ \begin{matrix} \dfrac{(-1)^s a_1^{ps+1}}{(p(p+2))^s . s!} \left\{ \begin{matrix} \dfrac{(p(s-1)+2)!_p}{2^s} & p \text{ odd} \\ \left(\dfrac{p}{2}(s-1)+1\right)!_{\frac{p}{2}} & p \text{ even} \end{matrix} \right\}, & \text{if} \quad i=1 \\ 0 & \text{if} \quad i \neq 1 \end{matrix} \right\}, \qquad (2.3.36)$$

where i = 1,2,3,...p for every s = 0,1,2,3,... .

Generally, $(r)!_p$ is a repeated factorial. It means that $(r)!_1=(r)!$, $(r)!_2=(r)!!$, $(r)!_3=(r)!!!$, and so on. Some of them appear in the special cases of p.

A sequence $\{a_n\}_{n=1}^{\infty}$ depends asymptotically on n of degree δ [6] and δ is given by

$$\delta = (\lambda - H + 1 - A) / (H-1), \qquad (2.3.37)$$

where λ is the degree of the polynomial $p(n)$, H is the highest order of the nonlinear term in the nonlinear equation for $\tilde{\phi}$ and $A = \sum_{i=1}^{H} \mu_i$ being the order of the derivative associated with each $\tilde{\phi}$ factor in the nonlinear term of the same equation. In this case, with equations (2.3.13) and (2.3.17), we can deduce that $\delta = \dfrac{2-p}{p}$ such that $\lambda = 2$ and $H = p + 1$ and $A = 0$.

Upon substituting from (2.3.36) into (2.3.14), we obtain

$$\tilde{\phi}(\xi) = \sum_{n=1}^{\infty} a_n g^n = \sum_{n=0}^{\infty} a_{pn+1} g^{pn+1} . \qquad (2.3.38)$$

i.e.

$$\tilde{\phi}(\xi)=a_1 g\sum_{n=0}^{\infty}\frac{(-1)^n (a_1 g)^{pn}}{n!(p\,(p+2))^n}\begin{Bmatrix}\dfrac{(p(n-1)+2)!_p}{2^n} & p \ \text{odd}\\ \left(\dfrac{p}{2}(n-1)+1\right)!_{\frac{p}{2}} & p \ \text{even}\end{Bmatrix}. \tag{2.3.39}$$

By using the binomial theorem (Gradshtyn and Ryzhik)[39], we have

$$\left(1+\frac{(a_1 g)^p}{(2\,(p+2))}\right)^{\frac{-2}{p}}=\sum_{n=0}^{\infty}\frac{(-1)^n (a_1 g)^{pn}}{n!(p\,(p+2))^n}\begin{Bmatrix}\dfrac{(p\,(n-1)+2)!_p}{2^n} & p \ \text{odd}\\ \left(\dfrac{p}{2}(n-1)+1\right)!_{\frac{p}{2}} & p \ \text{even}\end{Bmatrix}. \tag{2.3.40}$$

Then, equation (2.3.39) can be expressed in the following form

$$\tilde{\phi}(\xi)=\frac{(2p+4)^{\frac{1}{p}}dg}{\left(1+(dg)^p\right)^{\frac{2}{p}}}, \tag{2.3.41}$$

where

$$d=\frac{a_1}{(2p+4)^{\frac{1}{p}}}. \tag{2.3.42}$$

The convergence of the series in (2.3.40) or (2.3.41) requires that $|dg|<1$. This is motivated to, equation the choice $g_1=\frac{1}{g_2}=g=e^{-K\xi}$ only and then (2.3.41) holds for all $-\infty<\xi<\infty$. Now (2.3.41) is rewriting in the form

$$\tilde{\phi}(\xi)=\left(\frac{p+2}{2}\right)^{\frac{1}{p}}\left[\frac{2}{(dg)^{\frac{-p}{2}}+(dg)^{\frac{p}{2}}}\right]^{\frac{2}{p}}, \tag{2.3.43}$$

or

$$\tilde{\phi}(\xi)=\left(\frac{p+2}{2}\right)^{\frac{1}{p}}\left(\frac{2}{e^{\frac{p}{2}(K\xi+\Lambda)}+e^{\frac{-p}{2}(K\xi+\Lambda)}}\right)^{\frac{2}{p}}, \tag{2.3.44}$$

where $d = e^{-\Delta}$, and ($\Delta = -\ln d$) being an arbitrary phase shift. Then, it is easily deduced that

$$\tilde{\phi}(\xi) = \left(\frac{p+2}{2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}(K\,\xi + \Delta)\right). \qquad (2.3.45)$$

Back substitution, using the scale (2.3.12), we get

$$\hat{\phi}(\xi) = \left(\frac{v}{2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}(K\xi + \Delta)\right) \ . \qquad (2.3.46)$$

Also, using (2.3.9-a), (2.3.5) and (2.3.2), an exact solitary wave solution of the nonlinear (PDE) (2.3.1) is

$$u(x,t) = \left(\frac{v}{2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}\left(\sqrt{\frac{v}{\gamma}}(x - vt) + \Delta\right)\right). \qquad (2.3.47)$$

The solitary wave solution (2.3.47) is represented in three dimension plot (using mathematica 3.0 program) [9] for p = 1 and 4 as a kink solution in Figs [2.1] and [2.2] respectively where $v = 0.3$, $\gamma = 4.84 \times 10^{-4}$ and $\Delta = -6$ [40].

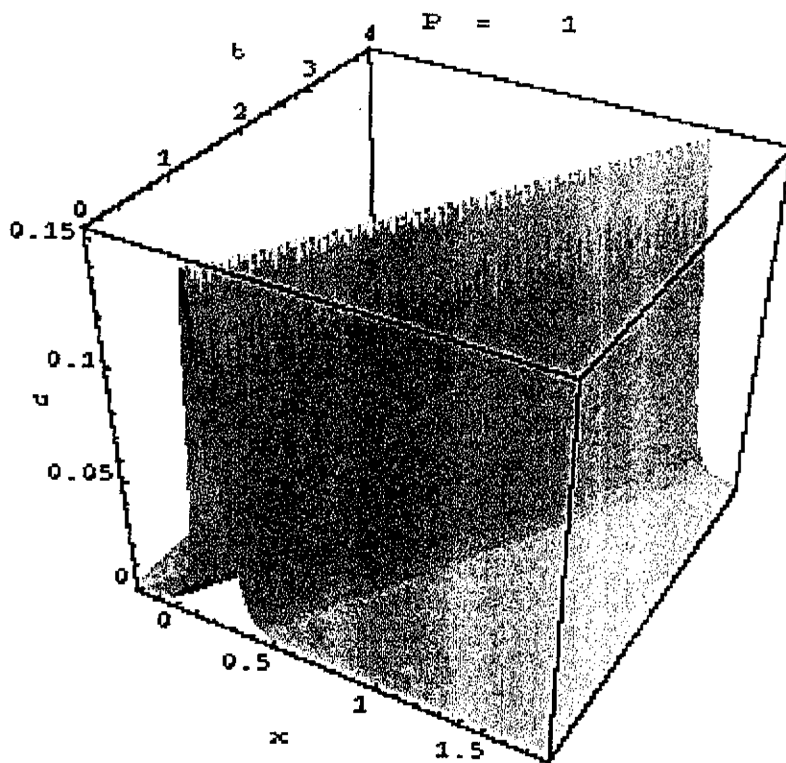


Fig (2.1) represents the solitary wave solution of the Generalized Kdv equation in three dimensions with

p = 1 (Kdv), $K = \sqrt{\frac{v}{\gamma}}$, v = 0.3, $\gamma = 4.84\times10^{-4}$, $\Delta = -6$, $c_1 = 0$, h = 0, u (x, t) = + (kink)

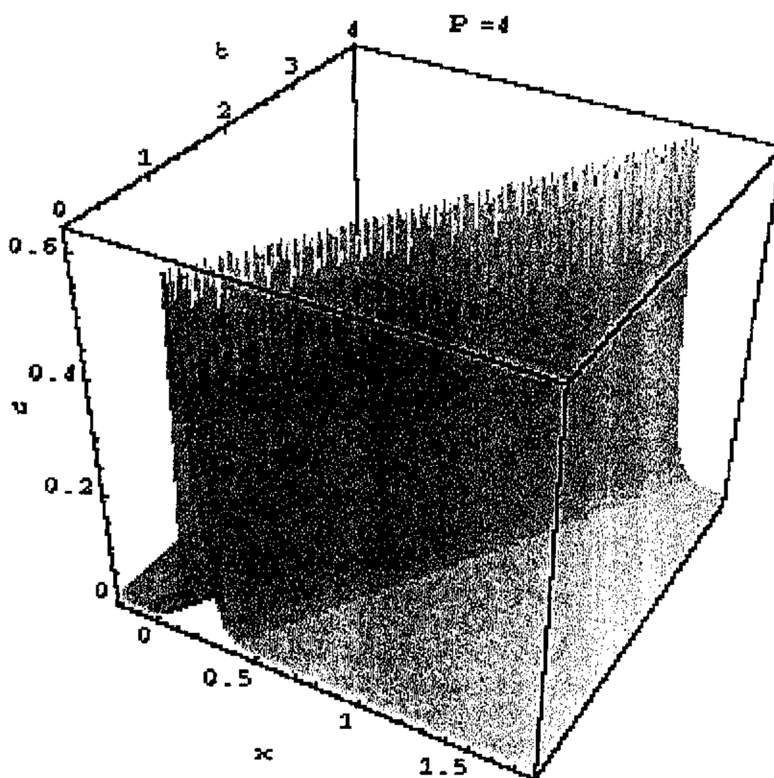


Fig (2.2) represents the solitary wave solution for Generalized Kdv equation in three dimensions with p = 4,

v = 0.3, $\gamma = 4.84\times10^{-4}$, $\Delta = -6$, $c_1 = 0$, h = 0, u (x, t) = + (kink).

Also the anti-kink solutions (in (2.3.47) for p even (i.e. with (-sign))) can be represented as example Fig [2.3] for p = 4 with the same values of v, γ and Δ as above .

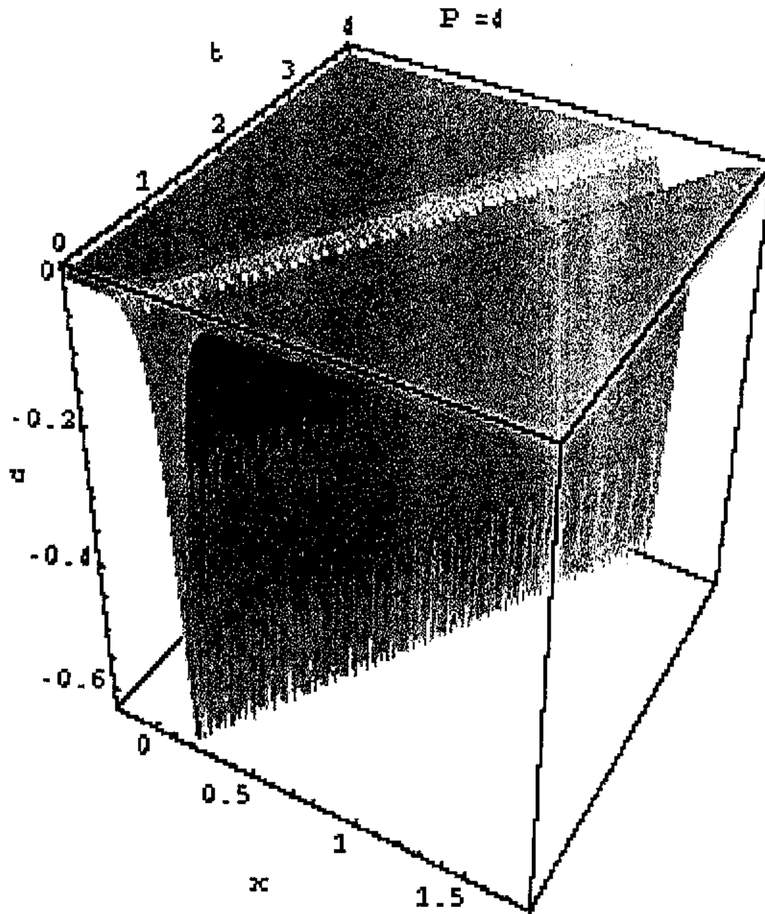


Fig (2.3) represens the solitary wave solution of the Generalized Kdv equation in three dimensions with p= 4, $K = \sqrt{\frac{v}{\gamma}}$, v = 0.3, $\gamma = 4.84\times10^{-4}$, $\Delta = -6$, $c_1 = 0$, h = 0, u(x, t) =-(antikink) solution.

As expected the travelling waves do not change in their shapes when the time increasing for each separately p. The solitary wave also narrow and sharp with increasing p, where the width is inverse proportional with the amplitude.

In The Previous Analysis When $c_1 = 0$ and $h = 0$, We Notice That

1. The general solution (2.3.47) at p = 1(namely the solution of the first equation (Kdv) in this class) is agreeing with the solitary wave solution of the Kdv equation $u_t + \alpha u u_x + u_{3x} = 0$. Reported in Hereman *et al* [6].

2. The solitary wave solution of the Mkdv equation $u_t + \alpha u^2 u_x + u_{3x} = 0$ in Hereman *et al* [6] is also agreeing with the corresponding one of the solution (2.3.47) (when p = 2). Notice that the relations $(p+1)(p+2) = \alpha$ (in each separation case) and $\gamma = 1$ in both cases for these comparison.

Another forms of solutions of Kdv and Mkdv equations depending on the choice of c_1 and h are deducing in the following sections of this chapter.

2.4 PARTICULAR SOLUTION OF THE KDV AND MKDV EQUATIONS

In this section, we treat the second case of the previous analysis, exactly equation (2.3.9-b) on the Kdv and Mkdv equations only. i.e. when (p = 1) and (p = 2) respectively in (2.3.1). That is going back to the (RR) with general p $(p \geq 3)$ has many non-linear terms of $\tilde{\phi}$ after scaling the nonlinear equation of $\hat{\phi}$. So, it is not easily

$$p\,v\hat{\phi}+(p+2)\left[\frac{(p+1)\,p}{2!}\left(\frac{v}{p+2}\right)^{1-\frac{1}{p}}\hat{\phi}^2+\frac{(p+1)\,p\,(p-1)}{3!}\left(\frac{v}{p+1}\right)^{1-\frac{2}{p}}\hat{\phi}^3+\frac{(p+1)\,p(p-1)\,(p-2)}{4!}\left(\frac{v}{p+2}\right)^{1-\frac{3}{p}}\hat{\phi}^4+\ldots\ldots\right]+\gamma\hat{\phi}_{2\xi}=0\ . \tag{2.4.1}$$

As mentioned above, equation (2.4.1) has p of the nonlinear terms if p is a positive integer, so that, we will discuss only the following two particular cases for a simplicity and the interested physics.

(1) The first case is the Kdv equation (2.3.18), the equations (2.3.9- b) and (2.4.1) give

$$c_1=\frac{v}{3},\qquad K^2=\frac{-v}{\gamma}, \tag{2.4.2}$$

and

$$v\hat{\phi}+3\hat{\phi}^2+\gamma\hat{\phi}_{2\xi}=0\ . \tag{2.4.3}$$

A similar treatment as in the previous section (2.2) requires that, the linear part of equation (2.4.3) has the solution in the form $e^{\pm K\xi}$. Now, we introduce a scaling parameter η such that

$$\hat{\phi}=\eta\,\tilde{\phi}\ , \tag{2.4.4}$$

and select $\eta=-\frac{v}{3}$, equation (2.4.3) gives

$$v\tilde{\phi}-v\tilde{\phi}^2+\gamma\tilde{\phi}_{2\xi}=0\ . \tag{2.4.5}$$

Upon defining $\tilde{\phi}$ in an infinite series expansion as:

$$\tilde{\phi}(\xi)=\sum_{n=1}^{\infty}a_n g^n(\xi)\ , \tag{2.4.6}$$

with $g_1=\frac{1}{g_2}=g(\xi)=e^{-K\xi}$, where $K_1=-K_2=K=\left(-\frac{v}{\gamma}\right)^{\frac{1}{2}}$, and substituting into equation (2.4.5), to yield

$$v\sum_{n=1}^{\infty} a_n g^n - v\left(\sum_{n=1}^{\infty} a_n g^n\right)^2 - v\sum_{n=1}^{\infty} a_n n^2 g^n = 0. \qquad (2.4.7)$$

Applying Cauchy's rule [39] on the non-linear term in the previous equation, we obtain the following (RR)

$$\left(n^2 - 1\right)a_n + \sum_{m=1}^{n-1} a_{n-m} a_m = 0, \quad n \geq 2, \delta = \frac{1}{2}. \qquad (2.4.8)$$

The (RR) (2.4.8) is identically agreeing with the corresponding one (2.3.19) when $c_1 = 0$. Consequently, the general coefficient a_n due to solving (RR) (2.4.8) is given by equation (2.3.20) and the solution $\widetilde{\phi}$ (using equation (2.3.45)) is

$$\widetilde{\phi} = \frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}(K\xi + \Delta)\right). \qquad (2.4.9)$$

Substituting from (2.4.9) into (2.4.4) where $\eta = \frac{-v}{3}$, we have

$$\hat{\phi} = -\frac{v}{2}\operatorname{sech}^2\left(\frac{1}{2}(K\xi + \Delta)\right). \qquad (2.4.10)$$

Consequently, the general solution is written as

$$u(x,t) = \frac{v}{3}\left[1 - \frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}(K(x - vt) + \Delta)\right)\right], \qquad (2.4.11)$$

where $u(x,t) = \phi(\xi) = c_1 + \hat{\phi}$, $c_1 = \frac{v}{3}$, $K^2 = -\frac{v}{\gamma}$ and $\Delta = -\ln d$ $(d = \frac{a_1}{6})$.

The solution (2.4.11) for the Kdv equation is shown in Figs [2.4] and [2.5] for

$v = -0.3$, $\gamma = 4.84 \times 10^{-4}$, $\Delta = -6$ and

$v = 0.3$, $\gamma = -4.84 \times 10^{-4}$, $\Delta = -6$.

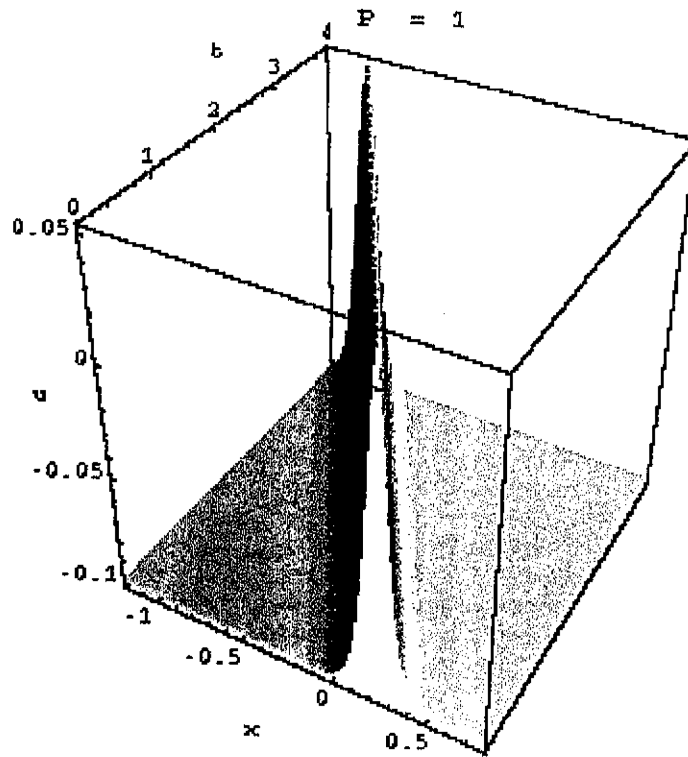


Fig (2.4) represents the solitary wave solution for the Kdv equation (p = 1) in three dimensions with v =- 0.3, $K=\sqrt{\frac{-v}{\gamma}}$, $\gamma = 4.84\times10^{-4}$, $\Delta=-6$, $c_1=\frac{v}{3}$, h = 0 ,u (x, t) = + (kink)

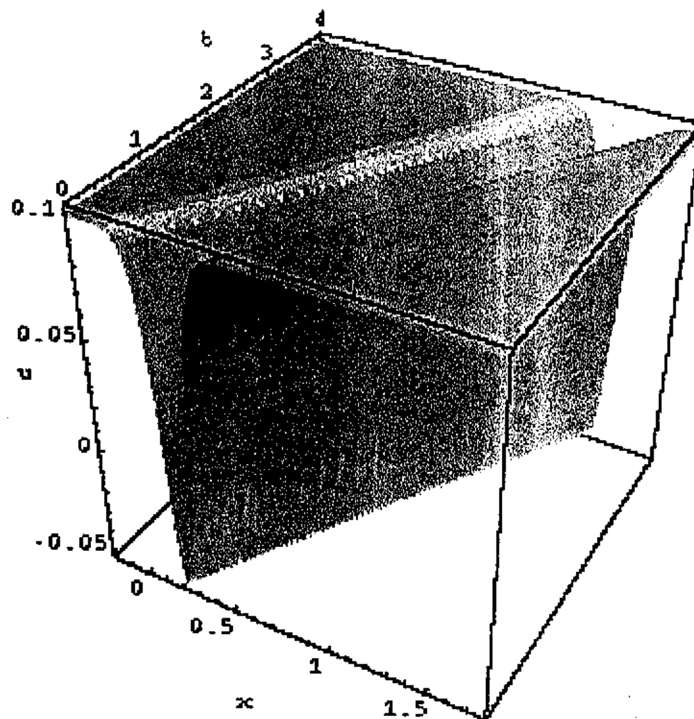


Fig (2.5) represents the solitary wave solution for the Kdv equation (p=1) in three dimensions with $v = 0.3$, $K=\sqrt{\frac{-v}{\gamma}}$, $\gamma = -4.84\times10^{-4}$, $\Delta=-6$, $c_1=\frac{v}{3}$, h = 0, u (x, t) = - (antikink)

(2) A second important example when ($c_1 \neq 0$) is the Modified kdv equation (2.3.21). So, equations (2.3.9 -b) and (2.4.1) become

$$c_1^2 = \frac{v}{4}, \qquad K^2 = -\frac{2v}{\gamma}, \tag{2.4.12}$$

and

$$2v\hat{\phi} \pm 6v^{\frac{1}{2}}\hat{\phi}^2 + 4\hat{\phi}^3 + \gamma\,\hat{\phi}_{2\xi} = 0, \tag{2.4.13}$$

where the linear part of equation (2.4.13) has the solution $e^{\pm K(v)\xi}$ and the real wave number K (equation (2.4.12)) requires that $\gamma > 0$ when $v < 0$ and $\gamma < 0$ if $v > 0$. Now, if we put $\hat{\phi} = \eta\,\tilde{\phi}$ into equation (2.4.13), we have

$$2v\tilde{\phi} \pm 6v^{\frac{1}{2}}\eta\tilde{\phi}^2 + 4\eta^2\tilde{\phi}^3 + \gamma\tilde{\phi}_{2\xi} = 0. \tag{2.4.14}$$

We can choose

$$\eta = \mp\frac{v^{\frac{1}{2}}}{3}, \tag{2.4.15}$$

then, equation (2.4.14) rewrites as

$$v\tilde{\phi} \mp v\tilde{\phi}^2 + \frac{2}{9}v\tilde{\phi}^3 + \frac{\gamma}{2}\tilde{\phi}_{2\xi} = 0 \ . \tag{2.4.16}$$

Using the equations (2.4.6) and (2.4.12) and applying Cauchy's rule on the nonlinear terms of equation (2.4.16), the following (RR) is obtained.

$$\left(n^2-1\right)a_n + \sum_{m=1}^{n-1} a_{n-m}a_m - \frac{2}{9}\sum_{l=2}^{n-1}\sum_{m=1}^{l-1} a_{n-l}a_{l-m}a_m = 0, \qquad n \geq 3, \quad \delta = 0 \tag{2.4.17}$$

where the coefficient a_1 is arbitrary and the first few coefficients (which are calculated using mathematica 3.0 program [9] see Appendix (B)) are given by

$$a_2 = \frac{-a_1^2}{3}, \quad a_3 = \frac{a_1^3}{9}, \quad a_4 = \frac{-a_1^4}{27}, \quad a_5 = \frac{-a_1^5}{81}, \ldots\ldots\ldots\ldots$$

We notice that the (RR) in this case (2.4.17) is similar to the corresponding one for combining equation of Kdv and Mkdv in Coffey [26] at $c_1 = 0$. Also, it is a greening with the (RR) for the nonlinear Klein Gordan equation at the case $c_1 \neq 0$ as in Hereman *et al* [6]. Consequently, the general coefficient a_n is easily deduced in the form:

$$a_{n+1} = (-1)^n \frac{a_1^{n+1}}{3^n}, \qquad \text{for} \quad n = 0,1,2,3,........ \tag{2.4.18}$$

Substituting from (2.4.18) into (2.4.6), we obtain

$$\tilde{\phi}(\xi) = \sum_{n=1}^{\infty} a_n g^n = \sum_{n=0}^{\infty} \frac{(-1)^n (a_1 g)^{n+1}}{3^n} , \tag{2.4.19}$$

where

$$g_1 = \frac{1}{g_2} = g \quad = e^{-K\xi} \text{ and } K_1 = -K_2 = K = \sqrt{\frac{-2v}{\gamma}}.$$

By using the binomial theorem [39], we get

$$\frac{1}{\left[1 + \frac{a_1 g}{3}\right]} = \sum_{n=0}^{\infty} \frac{(-1)^n (a_1 g)^n}{3^n}. \tag{2.4.20}$$

Then $\tilde{\phi}$ can be re-expressed as

$$\tilde{\phi}(\xi) = \frac{3dg}{1+dg},$$

where $d = \frac{a_1}{3}$. The convergence of (2.4.20) implies that $|dg| < 1$ and it is valid for all $-\infty < \xi < \infty$. Consequently, we can write

$$\tilde{\phi}(\xi) = 3 \left[\frac{e^{-K\xi - 2\Delta}}{1 + e^{-K\xi - 2\Delta}}\right] , \tag{2.4.21}$$

where $\Delta = -\frac{1}{2} \ln d$ (the phase shift).

It is easily deduced as

$$\tilde{\phi}(\xi)=\frac{3}{2}\left[1-\tanh\left(\frac{K\xi}{2}+\Delta\right)\right] \quad , \tag{2.4.22}$$

by using (2.4.15), we have

$$\hat{\phi}(\xi)=\mp\frac{\sqrt{v}}{2}\left[1-\tanh\left(\frac{K\xi}{2}+\Delta\right)\right], \tag{2.4.23}$$

where $\phi=c_1+\hat{\phi}$ and $c_1=\pm\frac{\sqrt{v}}{2}$, the general solution of (2.3.21) is

$$u(x,t)=\pm\frac{\sqrt{v}}{2}\left[\tanh\left(\frac{K}{2}(x-vt)+\Delta\right)\right]. \tag{2.4.24}$$

The solitary wave solution (2.4.24) for the Mkdv equation when $c_1\neq 1$ and $h = 0$ is shown in Figs [2.6] and [2.7] where

$v=0.3,\quad \gamma=-4.84\times10^{-4}\quad,\qquad \Delta=-6$ and

$v=0.3,\quad \gamma=-4.84\times10^{-4}\quad,\qquad \Delta=-6.$

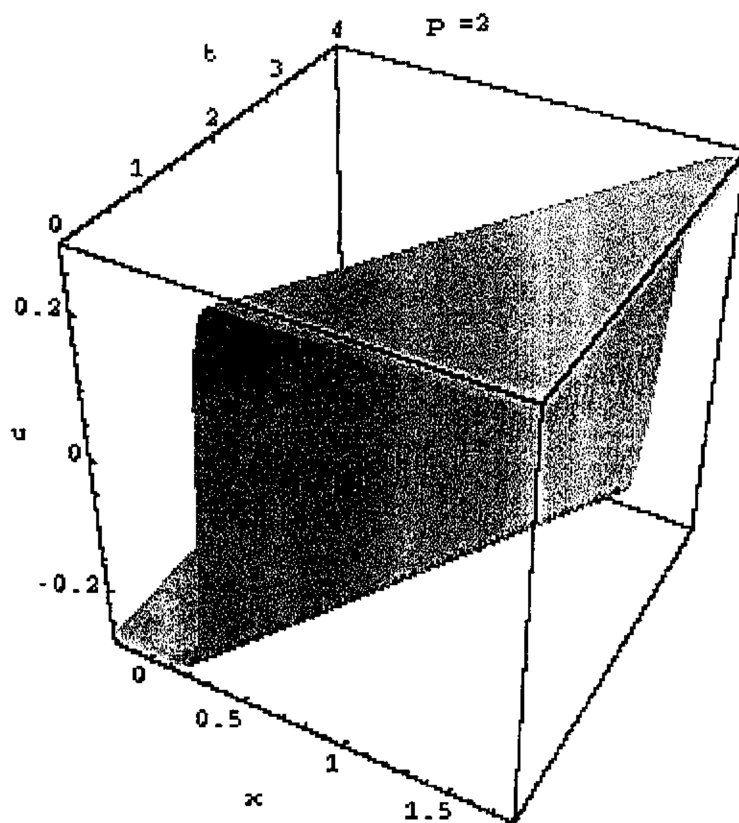


Fig (2.6) represents the solitary wave solution for the Mkdv equation (p = 2) in three dimension with $K=\sqrt{\frac{-2v}{\gamma}}$, $v = 0.3$, $\gamma = -4.84\times10^{-4}$, $c_1 = +\frac{\sqrt{v}}{2}$, $h = 0$, u (x, t)= +(kink)

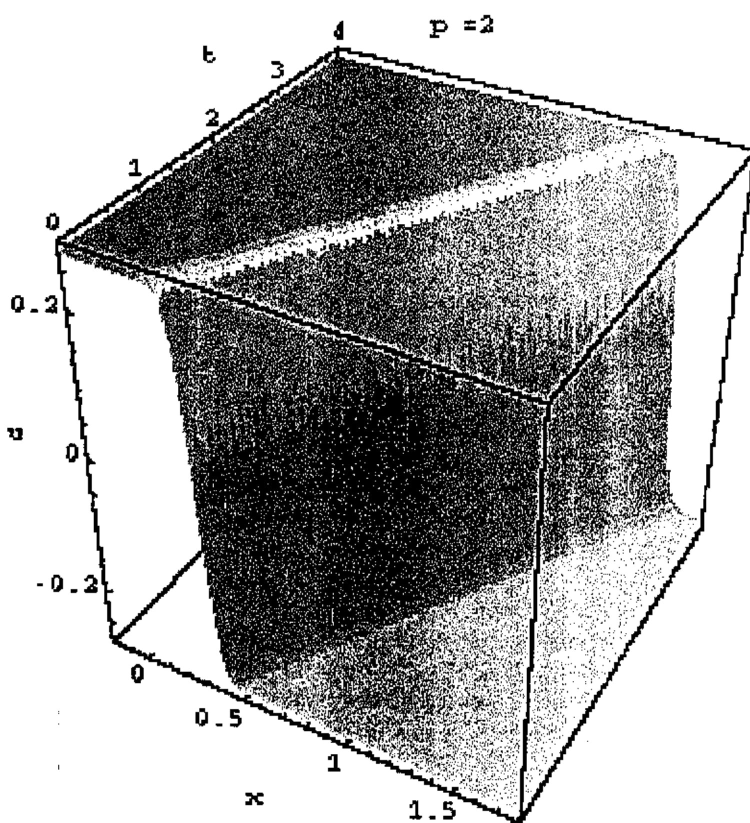


Fig (2.7) represents the solitary wave solution for the Mkdv equation (p = 2) in three dimension with $K=\sqrt{\frac{-2v}{\gamma}}$, $v = 0.3$, $\gamma = -4.84\times10^{-4}$, $\Delta = -6$, $c_1 = -\left(\frac{\sqrt{v}}{2}\right)$, $h = 0$, u (x, t)= -(antikink)

2.5 OTHER SOLUTIONS OF THE KDV EQUATION FOR SPECIAL CASES

Hereman *et al* [6] deduced a solution of the Kdv equation $u_t + \alpha u u_x + u_{3x} = 0$ in the form $\frac{3v}{2\alpha}\tanh^2\left(\frac{1}{2}\sqrt{\frac{-v}{2}}(x - vt) + \delta\right)$ such that $K = \pm\sqrt{\frac{-v}{2}}$ and $c_1 = \frac{3v}{2\alpha}$ with the same (RR) of the corresponding same equation (2.3.18) when $c_1 = 0$ and $h = 0$. Although, the above analysis of the Kdv equation (section (2.3) and (2.4)) does not contain this solution, we will study it to obtain the corresponding solution in this section. In this aim, we would like to refer that this solution depending on other choice of the parameter h (the constant of integration) i.e. $h \neq 0$. This choice go backs to the value of c_1 which does not appeared with $h = 0$. Consequently, reconsidering equation (2.3.1) with the main technique of the series expansion method and $p = 1$ or directly from equation (2.3.18)), equation (2.3.6-b) has the form

$$\left(-v c_1 + 3 c_1^2 - h\right) + \left(-v + 6 c_1\right)\hat{\phi} + 3\hat{\phi}^2 + \gamma\,\hat{\phi}_{2\xi} = 0. \qquad (2.5.1)$$

In order to the linear part of equation (2.5.1) has the solution $e^{\pm K\xi}$, the constant term $\left(3c_1^2 - vc_1 - h\right)$ must be equal zero. i.e.

$$c_1 = \frac{v \pm \sqrt{v^2 + 12h}}{6}. \qquad (2.5.2)$$

If we compare the coefficients of the studied Kdv equation (2.3.18) and the corresponding one [6], we get $\alpha = 6$ and $\gamma = 1$, then

$$c_1 = \frac{3\,v}{2\alpha} = \frac{3\,v}{12} = \frac{v}{4}$$

According to the used parameters, we get

$$h = \frac{-v^2}{16} \qquad (2.5.3)$$

Now, if we reconsider the condition $3c_1^2 - vc_1 + \frac{v^2}{16} = 0$, we cannot only obtain one value $c_1 = \frac{v}{4}$ but also another value $c_1 = \frac{v}{12}$, which did not studied in [6].

In the following, we will consider the different cases of c_1 and generally the wave number is

$$K^2 = \frac{v - 6c_1}{\gamma} \tag{2.5.4}$$

1- Firstly, if $c_1 = \frac{v}{4}$ then

$$K^2 = \frac{-v}{2\gamma}, \tag{2.5.5}$$

and equation (2.5.1) can be expressed as

$$v\hat{\phi} + 6\,\hat{\phi}^2 + 2\gamma\hat{\phi}_{2\xi} = 0\ . \tag{2.5.6}$$

If we choose the scaling $\eta = \frac{-v}{6}$, we will introduce

$$\hat{\phi} = \frac{-v}{6}\tilde{\phi}. \tag{2.5.7}$$

Upon using (2.5.7), equation (2.5.6) rewrites as

$$v\tilde{\phi} - v\,\tilde{\phi}^2 + 2\gamma\tilde{\phi}_{2\xi} = 0\ . \tag{2.5.8}$$

According to the main technique in the previous sections, let us put

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n, \tag{2.5.9}$$

where $g_1 = \frac{1}{g_2} = g \ = e^{-K\xi}$ and $K_1 = -K_2 = K = \sqrt{\frac{-v}{2\gamma}}$.

Apply the Cauchy's rule on the resulting equation, the following (RR) yields

$$\left(n^2 - 1\right)a_n g^n + \sum_{m=1}^{n-1} a_{n-m} a_m = 0, \qquad n \geq 2\ , \delta = 0, \tag{2.5.10}$$

and a_1 is arbitrary.

Notice that, the (RR) in equation (2.5.10) is the same (RR) (2.3.19) for the Kdv equation corresponding to $c_1 = 0$ and h = 0 (section (2.3)). Consequently $\tilde{\phi}(\xi)$ is easily deduce as similar as (2.3.45) to give

$$\tilde{\phi}(\xi)=\frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}(K\xi+\Delta)\right), \tag{2.5.11}$$

using the scale in (2.5.7), we get

$$\hat{\phi}(\xi)=\frac{-v}{4}\operatorname{sech}^2\left(\frac{1}{2}(K\xi+\Delta)\right). \tag{2.5.12}$$

Back substituting as indicated in section (2.3), such that

$$u(x,t)=\phi(\xi)=c_1+\hat{\phi}(\xi) \text{ and } c_1=\frac{v}{4},$$

we have

$$u(x,t)=\frac{v}{4}\tanh^2\left(\frac{1}{2}(K(x-vt)+\Delta)\right), \tag{2.5.13}$$

where $K^2=\frac{-v}{2\gamma}$ and $\Delta=e^{-\ln d}, d=\frac{a_1}{6}$.

The solution (2.5.13) for the Kdv equation represents in Figs [2.8] and [2.9] where

$v=-0.3$, $\gamma=4.84\times10^{-4}$, $\Delta=-6$ and

$v=0.3$, $\gamma=-4.84\times10^{-4}$, $\Delta=-6$.

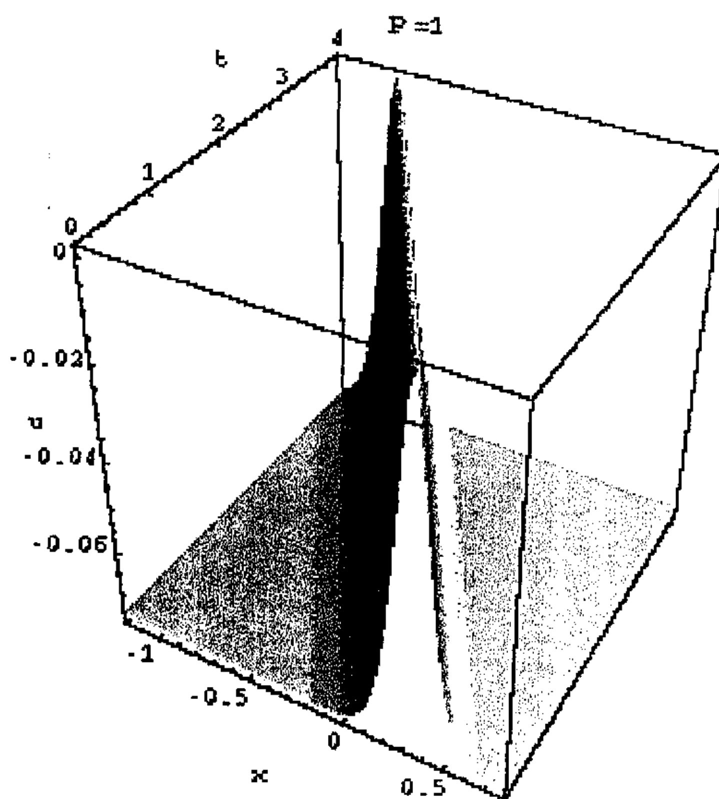


Fig (2.8) represents the solitary wave solution for the Kdv equation (p = 1) in three dimensions with v =- 0.3, $\gamma = 4.84\times10^{-4}$, $\Delta = -6, c_1 = \frac{v}{4}$, $h = \frac{-v^2}{16}, k = \sqrt{\frac{-v}{2\gamma}}$, u (x, t) = +(kink)

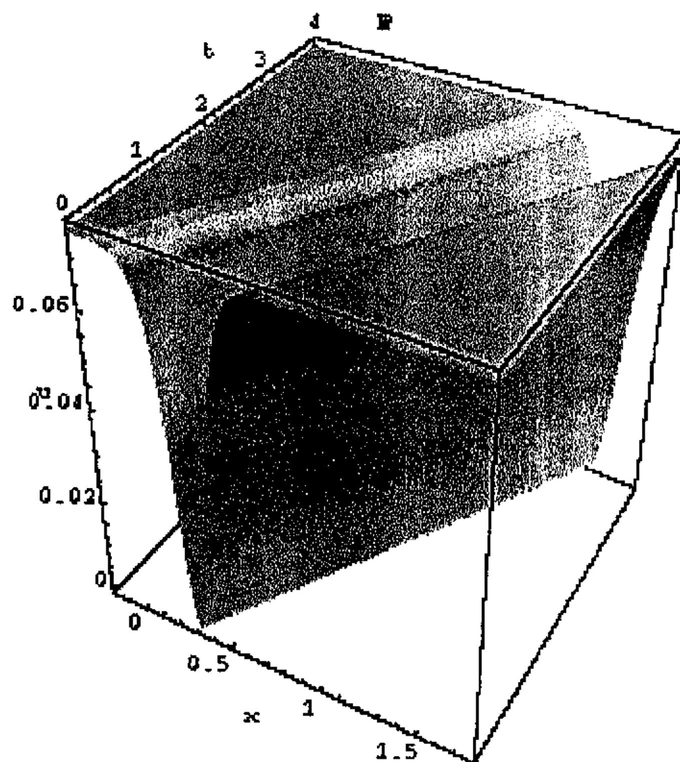


Fig (2.9) represents the solitary wave solution for the Kdv equation (p = 1) in three dimensions with v = - 0.3, $\gamma = 4.84\times10^{-4}$, $\Delta = -6, c_1 = \frac{v}{4}, h = \frac{-v^2}{16}, K = \sqrt{\frac{-v}{2\gamma}}$, u(x,t)= -(antikink)

2-Under the second choice of $c_1 = \frac{v}{12}$, equations (2.5.4) and (2.5.1) give $K^2 = \frac{v}{2\gamma}$ and

$$-v\hat{\phi} + 6\,\hat{\phi}^{2} + 2\gamma\hat{\phi}_{2\xi} = 0\,. \tag{2.5.14}$$

After re-scaling the coefficients of (2.5.14) by setting $\hat{\phi} = \eta\,\tilde{\phi}$ and choosing

$$\eta = \frac{v}{6}, \tag{2.5.15}$$

it becomes

$$-v\tilde{\phi} + v\,\tilde{\phi}^{2} + 2\gamma\tilde{\phi}_{2\xi} = 0\,. \tag{2.5.16}$$

Applying the main technique by using (2.5.9) and the Cauchy's product rule, we can obtain the same (RR) (2.5.10) or (2.3.19). Then, the solution can be deduced in the form (2.5.11). Using the scaling (2.5.15), we have

$$\hat{\phi} = \frac{v}{4}\operatorname{sech}^2\left(\frac{1}{2}(K\xi + \Delta)\right)\,. \tag{2.5.17}$$

Consequently, $\phi(\xi) = c_1 + \hat{\phi}(\xi)$, then the solution of the Kdv equation is

$$u(x,t) = \frac{v}{12}\left[1 + 3\operatorname{sech}^2\left(\frac{1}{2}\left(K(x - vt) + \Delta\right)\right)\right]\,, \tag{2.5.18}$$

where $K^2 = \frac{v}{2\gamma}$ and $\Delta = e^{-\text{Ind}}, d = \frac{a_1}{6}$.

The solution (2.5.18) for the Kdv equation where $h \neq 0, c_1 = \frac{v}{12}$ represents in Figs [2.10] and [2.11] when

$$v = 0.3\ , \quad \gamma = 4.84\times10^{-4}\ , \quad \Delta = -6 \quad \text{and}$$
$$v = -0.3\ , \quad \gamma = -4.84\times10^{-4}\ , \quad \Delta = -6.$$

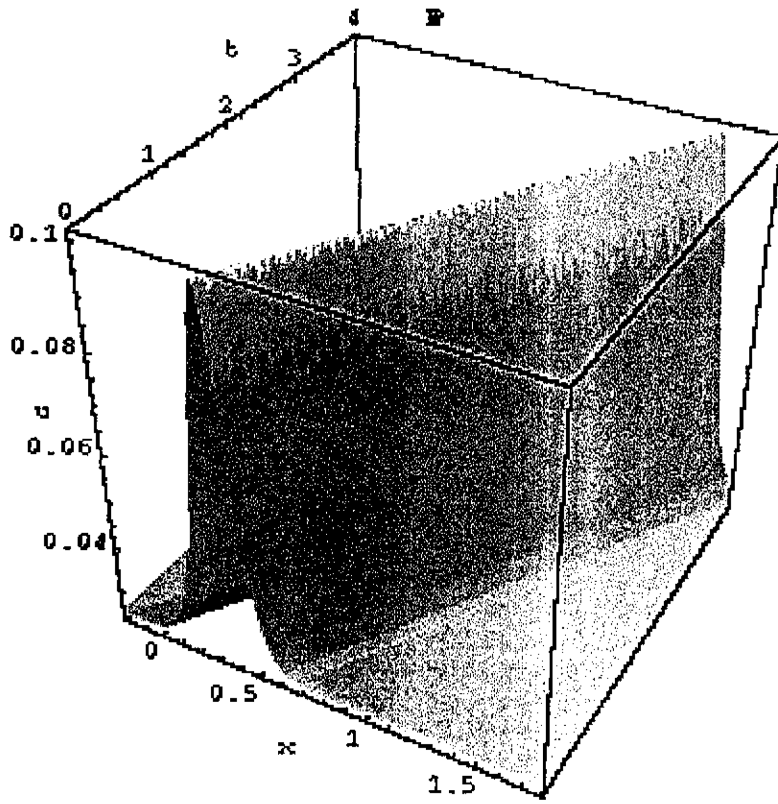


Fig (2.10) represents the solitary wave solution for the Kdv equation (p = 1) in three dimensions with $v = 0.3$, $\gamma = 4.84\times10^{-4}$, $\Delta = -6, c_1 = \frac{v}{12}$, $h = \frac{-v^2}{16}, K = \sqrt{\frac{v}{2\gamma}}$, u (x, t) =+(kink)

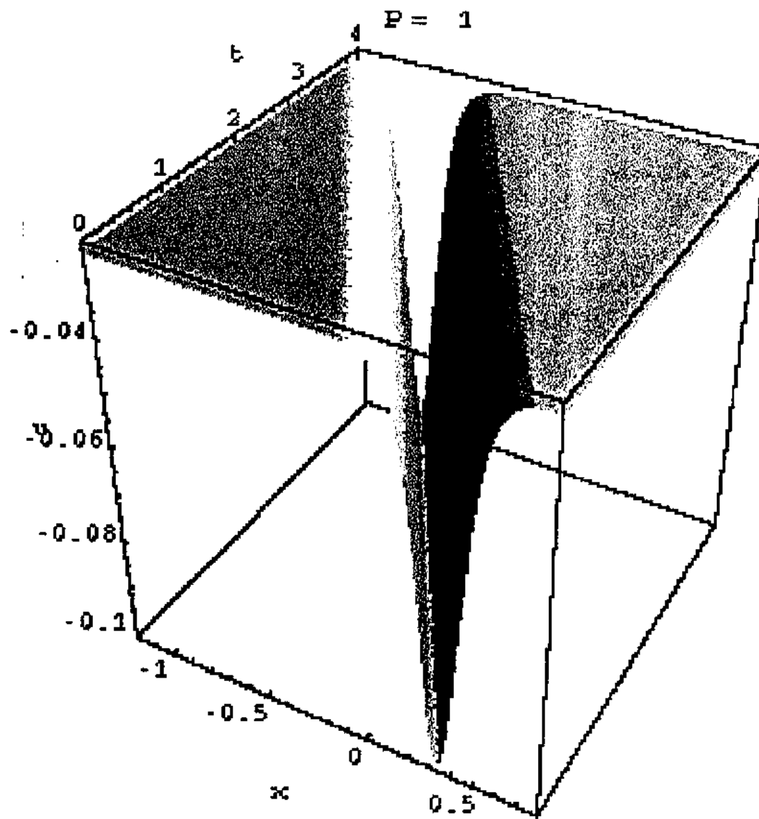


Fig (2.11) represents the solitary wave solution for the Kdv equation (p = 1) in three dimensions with $v = -0.3$, $\gamma = -4.84\times10^{-4}$, $\Delta = -6, c_1 = \frac{v}{12}$, $h = \frac{-v^2}{16}, K = \sqrt{\frac{v}{2\gamma}}$, u (x,t)=-(antikink)

2.6 CONCLUSION

From the above solutions of the equations (Kdv, Mkdv and Generalized Kdv), it is clear that the single travelling solutions to a nonlinear wave equation could be obtained by a trial procedure of the series method with simple algebraic calculations. It could also be seen that the present method may be generalized to solve several cases of nonlinear equations, which used in physical sciences.

Through the presentations of various kinds of equations, we may achieve the following concluding remarks.

1- The present approach only involves algebraic calculations, which is much easier than the differential and integral derivations, compared with other methods, where these calculations used to find the recursion relation (RR) for each cases of our problem.

2- It may be possible to apply the series method to obtain solutions for the Generalized Kdv equation when $c_1 \neq 0$ and the constant of integration h not equal zero for $P > 2$.

3- By introducing a series expansion method, numerous of other examples can be treated, as well as a coupled non-linear wave equation system.

4- Our main purpose was to show that the present method is quite a powerful tool for obtaining exact analytical solutions to higher order of NPDE.

Chapter(3)

CHAPTER (3)

SOME OF FINITE DIFFERENCE METHODS FOR SOLVING THE GENERALIZED KDV EQUATION

3.1 INTRODUCTION

Numerical analysis is still a common tool for the study of partial differential equations. It is convenient also in the class of nonlinear equations. The restrictive factors of the used numerical method are the final accuracy of the required results and the limitations of time. The later factor is how computing a few interested with the appearing the high-speed technology computers.

One of the most popular techniques, which is used to solve the nonlinear partial differential equations, is the finite difference method, which is an essential part of this chapter. The finite difference scheme is generally accreted by some factors such as consistency, stability and truncation error.

Various numerical methods applied to the Kdv equation, among those (the finite difference methods), i.e. the explicit scheme of Zabusky & Kruskal [7], and some of the implicit methods such as Hopscotch method [8], a scheme due to Goda, a scheme suggested by M. Kruskal and a scheme developed using notions of the inverse scattering transform by R.Thiab and M.J.Ablowitz [41]. Some of the previous schemes applied also on the Mkdv equation by R.Thiab and M.J.Ablowitz [41].

We propose to develop some of these schemes for applying on the studied problem (Generalized Kdv equation). Consequently in this chapter, we construct two finite difference schemes for the numerical solution of Generalized Kdv equation. The first choice is the classical Zabusky and Kruskal method and the other is the Hopscotch method.

The plan of the work in this chapter is the following:

As mentioned the major part of this chapter is reported to development two methods for solving the Generalized Kdv equation (1.2.1) numerically as:

1. In section (3.2), the finite difference forms of u_t, u_x and u_{3x} are defined as a notation in order to introduce them in our problem.

2. In sections (3.3) and (3.4), the numerical methods are described for solving the Generalized Kdv equation (1.2.1) numerically. The formula for each scheme is deduced, and the truncation error for each method is calculated (see Appendix (A)), also the stability condition for each method is illustrated.

3. In section (3.5), the numerical solutions for p = 1,2,3,4,5,6 with the Generalized Kdv equation are computed for each method by writing two Fortran programs with $v = 0.3$ and $\gamma = 4.84 \times 10^{-4}$ at finite value of time. The numerical results for each method are described with some improvements are given in details. Also, through the presentation of results, lists of tables are shown to compare the accuracy between the two methods using L_∞ error.

From the comparison of numerical and analytical solutions of the Generalized Kdv equation for each cases of p, which are plotted by using (mathematica 3.0 program) [9], we notice that the implicit (Hopscotch) method is more accurate than the explicit Zabusky and Kruskal method when both work.

3. 2 DEFINITIONS AND NOTATIONS

According to the usual finite difference notation, we define

$$u(x_n, t_m) = u(nh,\ m\tau) = u_n^m \ , \tag{3.2.1}$$

to be the difference approximation for u at the grid point $(nh, m\tau)$, where h is the spatial step length and τ is the temporal step length.

The general form of the Generalized Kdv equation (1.2.1) is solved numerically, consequently, it is required to express u_t, u_x and u_{3x} in their finite difference forms as follows:

$$\frac{\partial}{\partial t}u_n^m = \frac{1}{2\tau}H_t u_n^m = \frac{1}{2\tau}\left(u_n^{m+1} - u_n^{m-1}\right), \tag{3.2.2}$$

$$\frac{\partial}{\partial x}u_n^m = \frac{1}{2h}H_x u_n^m = \frac{1}{2h}\left(u_{n+1}^m - u_{n-1}^m\right), \tag{3.2.3}$$

$$\frac{\partial^3}{\partial x^3}u_n^m = \frac{1}{2h^3}H_x\delta_x^2 u_n^m = \frac{1}{2h^3}\left(u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m\right). \tag{3.2.4}$$

These expressions are of the second order accuracy.

3.3 THE METHOD OF ZABUSKY AND KRUSKAL

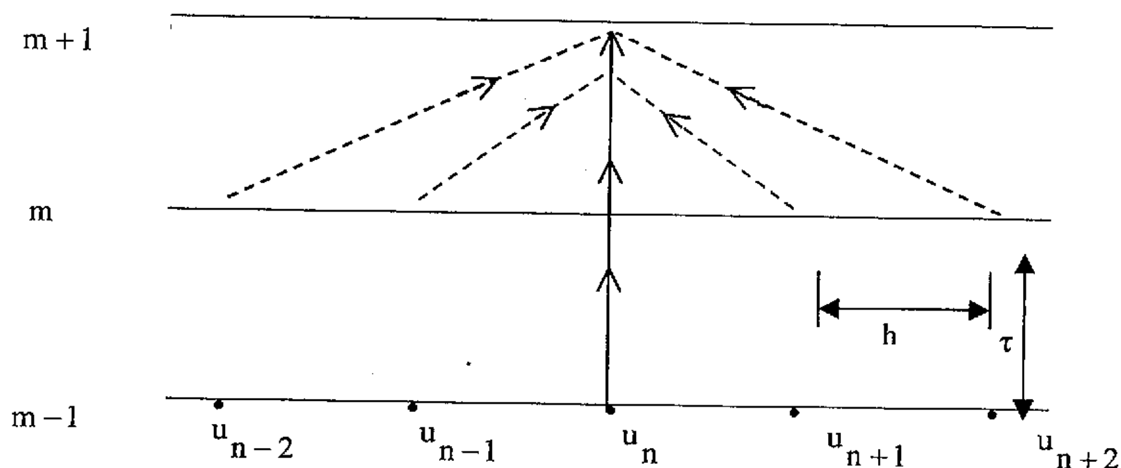
3.3-1 The Finite Difference Scheme and Its Truncation Error

In the original work of Zabusky and Kruskal [7] for Kdv equation, the explicit leapfrog finite difference scheme [42] is used. When it is applied to equation (1.2.1), the developed finite difference scheme is obtained in the following form:

$$u_n^{m+1} = u_n^{m-1} - \frac{(p+1)(p+2)r}{3^p}\left(u_{n+1}^m + u_n^m + u_{n-1}^m\right)^p\left(u_{n+1}^m - u_{n-1}^m\right) - \frac{\gamma r}{h^2}\left(u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m\right), \tag{3.3.1}$$

where $r = \frac{\tau}{h}$ is the ratio between the time and space increments.

It is clear that equation (3.3.1) is a scheme of three-time level. i.e. in order to obtain u_n at the time level $(m+1)$, we need the values of $u_{n-2}, u_{n-1}, u_{n+1}, u_{n+2}$ at the previous time level m, in addition to the value of u_n at the time level (m -1). The given figure illustrates how the method works.



To obtain the truncation error of the scheme (3.3.1), the advection speed u, in the nonlinear term $\left(u^{p}\right)$ of equation (1.2.1) is assumed to be locally constant as well as in the study of the stability analysis in the following section. Then the scheme can be written in the form

$$u_n^{m+1} - u_n^{m-1} + r(p+1)(p+2)\,\bar{u}^{p}\left(u_{n+1}^{m} - u_{n-1}^{m}\right) + \frac{\gamma r}{h^{2}}\left(\left(u_{n+2}^{m} - u_{n-2}^{m}\right) - 2\left(u_{n+1}^{m} - u_{n-1}^{m}\right)\right) = 0. \quad (3.3.2)$$

Upon applying the Taylor expansion, the truncation error of the scheme (3.3.1) or (3.3.2) is of order $\left\{0\,\left(\tau^{2}\right) + 0\,\left(h^{2}\right)\right\}$ as indicated in Appendix (A).

3. 3-2 The Stability Analysis

To study the stability analysis of the Generalized Kdv equation (1.2.1) using the scheme (3.3.1), we first linearize (1.2.1). This is done as mentioned above by assuming that the advection speed u, in the nonlinear term $u^{p}u_{x}$, is locally constant in equation (1.2.1), which takes the form

$$u_t + (p+1)\,(p+2)\,\bar{u}^{p}u_x + \gamma\,u_{3x} = 0. \quad (3.3.3)$$

Substitution equations (3.2.2), (3.2.3), and (3.2.4) into the above equation, the equation (3.3.1) takes the form

$$u_n^{m+1} = u_n^{m-1} - r\left((p+1)\,(p+2)\,\bar{u}^{p}H_x + \frac{\gamma}{h^{2}}H_x\delta_x^{2}\right)\,u_n^{m} \quad (3.3.4)$$

According to Zabusky and Kruskal scheme $\overline{u}^p$ will to be $\left(\frac{1}{3}\left(u_{n+1}^m + u_n^m + u_{n-1}^m\right)\right)^p$.

The stability analysis of Zabusky and Kruskal scheme (corresponding to the Kdv equation) is not given in their paper [7], only the stability condition is given in [8] and [41]. Here, we develop the stability analysis of the scheme (3.3.4) depending on the general approach of Von-Neumann [42] and the work of Miller [43]. We start by defining

$$v_n^m = u_n^{m-1} . \tag{3.3.5}$$

Then, we write (3.3.4) as a system of the following equation:

$$\left\{\begin{aligned} u_n^{m+1} &= v_n^m - r\left((p+1)(p+2)\,\overline{u}^p H_x + \frac{\gamma}{h^2} H_x \delta_x^2\right) u_n^m \\ v_n^{m+1} &= u_n^m \end{aligned}\right\} \tag{3.3.6}$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix}_n^{m+1} = \begin{bmatrix} -r\left((p+1)(p+2)\overline{u}^p H_x + \frac{\gamma}{h^2} H_x \delta_x^2\right) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_n^m \tag{3.3.7}$$

i.e.

$$w_n^{m+1} = A w_n^m , \tag{3.3.8}$$

where

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{3.3.9}$$

and

$$A = \begin{bmatrix} -r\left((p+1)(p+2)\overline{u}^p H_x + \frac{\gamma}{h^2} H_x \delta_x^2\right) & 1 \\ 1 & 0 \end{bmatrix} \tag{3.3.10}$$

The matrix A is called the amplification matrix. To apply the Von-Neuman stability analysis, we use the Fourier mode

$$w_n^m = w_o^m \exp\left[i\,\xi\,(nh)\right], \tag{3.3.11}$$

where w_o^m is a constant vector. Then, one can easily obtain

$$A=\begin{bmatrix} -2ir\sin(\xi h)\left[(p+1)(p+2)\bar{u}^p-\frac{4\gamma}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right] & 1 \\ 1 & 0 \end{bmatrix} \qquad (3.3.12)$$

As usual, the eigen values of A are obtained from the secular equation,

$$|A-\lambda I|=0, \qquad (3.3.13)$$

which gives

$$\lambda^2+2ir\sin(\xi h)\left[(p+1)(p+2)\bar{u}^p-\frac{4\gamma}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right]\lambda-1=0$$

The Von-Neumann stability condition implies that

$$|\lambda_{max}|\leq 1. \qquad (3.3.14)$$

Upon applying Miller's method, we write

$$f(z)=z^2+iBz-1,$$

where

$$B=2r\sin(\xi h)\left[(p+1)(p+2)\bar{u}^p-\frac{4\gamma}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right]$$

Suppose that $z^*=\bar{z}^{-1}$, $\bar{z}$ is the complex conjugate of z, then

$$f(z^*)=\bar{z}^{-2}+iB\bar{z}^{-1}-1,$$

and

$$\overline{f(z^*)}=z^{-2}-iBz^{-1}-1.$$

According to $f^*(z)=z^2\overline{f(z^*)}$, we have

$$f^*(z)=1-iBz-z^2.$$

Then, define the Bezout resultant [43]

$$\check{f}=(f^*(0)f(z)-f(0)f^*(z))/\ z$$

Hence, clearly $|f^*(0)|\not>|f(0)|$. For f to be Von-Neumann, we must have

i. $\check{f}=0$, and

ii. f' (the derivative of f w. r. to z) is Von-Neumann.

In this case

$$\check{f} = (z^2 + iBz - 1) + (1 - iBz - z^2) = 0.$$

and

$$f'(z) = 2z + iB$$

Then, we require that $|z| \le 1$ for the function $f'(z)$, i. e.

$$|z| = \left|\frac{-iB}{2}\right| \le 1.$$

By ensuring $|z| \le 1$, the resulting zeroes of $f(z)$ lie on the unit circle [43]. Therefore

$$r^2 \sin^2(\xi h)\left((p+1)(p+2)\bar{u}^p - \frac{4\gamma}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right)^2 \le 1 \qquad (3.3.15)$$

It is easy to deduce that the maximum value of the left-hand side in the previous inequality that occurs when

$$\frac{\xi h}{2} = \frac{\Pi}{3}$$

then

$$r^2\left(\frac{3\gamma}{h^2} - (p+1)(p+2)\bar{u}^p\right) \le \frac{4}{3} \quad .$$

Consequently, the scheme is conservative and the stability condition is

$$r\left|\frac{\gamma}{h^2} - \frac{(p+1)(p+2)}{3}\bar{u}^p\right| \le \frac{2}{3\sqrt{3}}, \qquad (3.3.16)$$

where $\bar{u}$ is the maximum value in the range of our investigation. The stability condition (3.3.16) is agreeing with the correspond one [41], when p = 1 (Kdv equation).

3. 4 THE HOPSCOTCH METHOD

3. 4-1 Theoretical Background

Rewrite the Generalized Kdv equation (2.3.1) as

$$u_t = Lu, \tag{3.4.1}$$

where the operator $L = -(p+1)(p+2)u^p \frac{\partial}{\partial x} - \gamma \frac{\partial^3}{\partial x^3}$.

Greig and Morris [8] proposed an appropriate discrete approximation to L, say L_h, so that equation (3.4.1) takes the form

$$u_t = L_h u + O(h^\sigma), \qquad \sigma > 0 \tag{3.4.2}$$

Ofcourse, the truncation error term $O(h^\sigma)$ is the difference between $L_h u$ and Lu. If we suppose that w (t) is a semi-discrete solution i. e.

$$w_t = L_h w, \qquad w(0) = u_0$$

Then,

$$w(t) = \exp\left[\int_{m\tau}^{(m+1)\tau} L_h(\psi)d\psi\right] w(t-\tau). \tag{3.4.3}$$

To obtain discrete in time approximations, we have to approximate both the integral in (3.4.3) and the resulting exponential. For the integral, we use the rectangular rule, i.e.

$$\int_{m\tau}^{(m+1)\tau} L_h(\psi)\, d\psi = \tau L_h(\underline{\psi}).$$

We make a truncation error $0(\tau^2)$, for $\underline{\psi} \neq \left(m + \frac{1}{2}\right)\tau$ while for $\underline{\psi} = \left(m + \frac{1}{2}\right)\tau$, the truncation error, corresponding to the resulting midpoint rule, is $0(\tau^3)$. For this situation $\underline{\psi} = m\tau$ is adequate.

Greig and Morris [8] proposed a (0.1) pade′ approximation for $\exp[\tau L_h(m\tau)]$ to have

$$v(t) = [I + \tau L_h(m\tau)]v(t-\tau),$$

and the truncation error is $\tau\, 0\left(\tau + h^{\sigma}\right)$, where I is the unit operator. Another way to write the above equation is

$$v_{m+1} = \left[I + \tau L_h(m\tau)\right] v_m \tag{3.4.4}$$

They followed Gourlay [44] in using equation (3.4.4) as the basis of a class of the Hopscotch method, namely,

$$v_{m+1} + \tau\,\theta_{m+1} L_h((m+1)\tau)\, v_{m+1} = v_m + \tau\,\theta_m L_h(m\tau)\, v_m, \tag{3.4.5}$$

where θ_m is the Hopscotch switch.

To define a Fourier solution of the Generalized Kdv equation, we have to take its linear form as equation (3.3.3) (in which $u^p u_x$ is replaced by $\bar{u}^p u_x$). The Fourier solution is

$$u = \sum_{k=-\infty}^{\infty} c_k \exp\, i\left(k\,x - \ell\,(k)\, t\right), \tag{3.4.6}$$

where k is the wave number and $\ell(k)$ is the frequency. The relation between $\ell(k)$ and k is given by the dispersion relation

$$\ell(k) = (p+1)\,(p+2)\,\bar{u}^{p}\, k - \gamma k^3. \tag{3.4.7}$$

The k<u>th</u> component of (3.4.6) is

$$u_k = c_k \exp i\left(k\,x - \ell\,(k)\, t\right) \tag{3.4.8}$$

Note, the similarity between this equation and (3. 3. 11). Then,

$$u_k\,(t+\tau) = c_k \exp i\left(kx - \ell(k)\,(t+\tau)\right)$$

$$= u_k(t)\, \exp\left(-i\ell\,(k)\,\tau\right). \tag{3.4.9}$$

This equation states that, the k<u>th</u> component of the solution moves a distance $\ell(k)\,\tau$, in time τ, in a direction equals to the sign of $\ell(k)$ with preserved amplitude. Consequently, if w_k is the k<u>th</u> component of the Fourier solution w of the difference approximation (3.4.4), then

$$w_k\left((m+1)\,\tau\right) = g(k)\, w_k\,(m\,\tau), \tag{3.4.10}$$

where $g\,(k)$ is called the amplification factor and is written as

$$g(k) = |g(k)| \exp i\phi \,, \tag{3.4.11}$$

where $\phi = \arg(g(k))$ is real.

It is clear that the application of difference operator L_h to u_k as defined by (3.4.8) gives a solution whose amplitude and phase are modified by $|g(k)|$ and ϕ respectively. If $|g(k)| = 1$, for all k, then the application of the difference operator dose not affect the amplitude of the solution, and such a scheme is called conservative or non-dissipative. The ratio-$\arg g(k)/\ell(k)\tau = R$, determines the phase error of the numerical method, which is used. If $0 < R < 1$, there is a phase lag of the numerical solution relative to the theoretical solution. If $R > 1$, it is the other way around. For R=1 the numerical solution gives exact phase.

3.4-2 The Hopscotch Finite Difference Scheme

The Generalized Kdv equation (1.2.1) with usual notations (3.2.2-4) and (3.4.4) gives

$$u_n^{m+1} = u_n^m - \frac{r}{2}(p+1)(p+2) H_x F_n^m - \frac{r\gamma}{2h^2} H_x \delta_x^2 u_n^m, \tag{3.4.12}$$

where

$$F_n^m = \frac{u^{p+1}|_n^m}{p+1}.$$

Clearly, (3.4.12) is an explicit form because the right hand side is given in terms of u^m. The implicit form is

$$u_n^{m+1} = u_n^m - \frac{r}{2}(p+1)(p+2) H_x F_n^{m+1} - \frac{r\gamma}{2h^2} H_x \delta_x^2 u_n^{m+1}, \tag{3.4.13}$$

where the right hand side is given in terms of u^{m+1}, u^m.

Then, the general form of the Hopscotch schemes that equivalent to (3.4.12) and (3.4.13), depending on [44] and [42] is

$$u_n^{m+1} + \theta_n^{m+1}\left[\frac{r}{2}(p+1)(p+2) H_x F_n^{m+1} + \frac{r\gamma}{2h^2} H_x \delta_x^2 u_n^{m+1}\right]$$

$$=u_n^m - \theta_n^m \left[\frac{r}{2} (p+1)(p+2) H_x F_n^m + \frac{r\gamma}{2h^2} H_x \delta_x^2 u_n^m \right], \qquad (3.4.14)$$

where

$$\theta_n^m = \begin{Bmatrix} 1 & \text{if} \quad n+m \quad \text{is even} \\ 0 & \text{if} \quad n+m \quad \text{is odd} \end{Bmatrix}$$

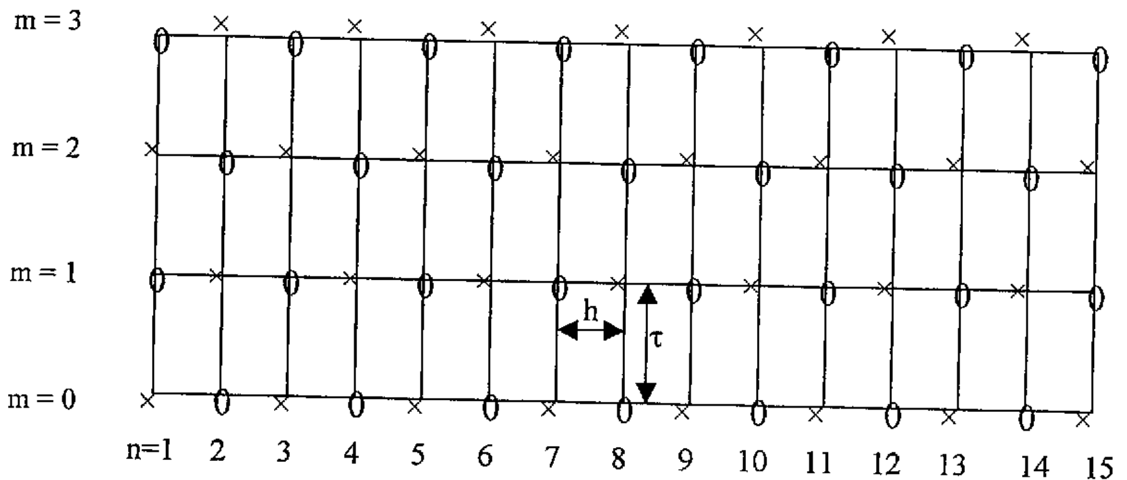
Rewriting equations (3.4.12) and (3.4.13) in terms of function values, we have

$$u_n^{m+1} = u_n^m - \frac{r}{2} (p+1)(p+2) \left[F_{n+1}^m - F_{n-1}^m \right] - \frac{r\gamma}{2h^2} \left[u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m \right], \qquad (3.4.15)$$

and

$$u_n^{m+1} = u_n^m - \frac{r}{2} (p+1)(p+2) \left[F_{n+1}^{m+1} - F_{n-1}^{m+1} \right] - \frac{r\gamma}{2h^2} \left[u_{n+2}^{m+1} - 2u_{n+1}^{m+1} + 2u_{n-1}^{m+1} - u_{n-2}^{m+1} \right]. \qquad (3.4.16)$$

Consequently, equation (3.4.15) is used at grid points for which $(n+m)$ is even and equation (3.4.16) is used at grid points for which $(n+m)$ is odd. The given figure illustrates how the calculations are carried out.



Explicit calculations	0	even point	Equation (3. 4. 15) is used
Implicit calculations	×	odd point	Equation (3. 4. 16) is used.

To be specific the work of the scheme, let us assume that m is even, then equation (3.4.15) is applied for the points n = 2,4,6...N-1, (circle in given figure) where the solution is sought in the rectangle

$$\left(0 \le x \le Nh\right) \times \left(t > 0\right) \qquad \text{and N is odd.}$$

Further, we assume that we can determine the solution at x = 0, – h, Nh and (N+1)h, either from given boundary conditions or by an appropriate extrapolation techniques [45]. Consequently, we assume that $u_0 = u_{-1} = u_N = u_{N+1} = 0$, for all t that is agreeing with the assumption of Greig and Morris [8] when they sought the Kdv problem by using the same scheme. The obtained solutions from equation (3.4.15) are used in equation (3.4.16). Hence, rearranging it to obtain

$$u_n^{m+1} + \frac{r\gamma}{2h^2}\left(u_{n+2}^{m+1} - u_{n-2}^{m+1}\right) = u_n^m - \frac{r}{2}(p+1)(p+2)\left(F_{n+1}^{m+1} - F_{n-1}^{m+1}\right)$$

$$+\frac{r\gamma}{h^2}\left(u_{n+1}^{m+1} - u_{n-1}^{m+1}\right). \qquad (3.4.17)$$

This algorithm, under the present assumption that m is even and N odd, is to be applied for n =1,3,5...N-2, (crosses in the figure). All the entries on the right hand side of (3.4.17) are known and we can write it as

$$u_n^{m+1} + \frac{r\gamma}{2h^2}\left(u_{n+2}^{m+1} - u_{n-2}^{m+1}\right) = K_n^m\,, \quad n = 1,3\ \ldots \text{N-2} \qquad (3.4.18)$$

where K_n^m is represented the right hand side of equation (3.4.17). Or, it can be put in the following matrix form

$$Au^{m+1} = K, \qquad (3.4.19)$$

where

$$
A = \begin{bmatrix}
1 & \frac{r\gamma}{2h^2} & 0 & 0 & \cdots & & 0 \\
-\frac{r\gamma}{2h^2} & 1 & \frac{r\gamma}{2h^2} & 0 & \cdots & & 0 \\
\vdots & & & & & & \\
\vdots & & & & -\frac{r\gamma}{2h^2} & 1 & \frac{r\gamma}{2h^2} \\
0 & \cdots & & & & -\frac{r\gamma}{2h^2} & 1
\end{bmatrix} \qquad (3.4.20)
$$

$$
u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ \vdots \\ u_{N-2} \end{bmatrix}^{m+1} \qquad (3.4.21)
$$

and $K = \left[K_1, K_3, \ldots\ldots \ldots\ldots, K_{N-2}\right]^T$. In general K_1 and K_{N-2} are suitably modified versions of the right hand side of equation (3.4.17), taking into account $\left(\frac{-\gamma r}{2h^2}\right)u_{-1}^{m+1}$ and $\left(\frac{\gamma r}{2h^2}\right)u_N^{m+1}$, respectively.

For m odd, we will obtain the obvious change of subscripts in the vectors u and K in equation (3.4.19), but the coefficient matrix A will remain unchanged.

For N even, we will obtain a similar matrix of coefficients, however, at alternate time levels, i.e. values of m, the order of the matrix will vary by one as the number of the unknowns corresponding to the implicit system (3.4.16), which is either $\frac{1}{2}(N-1)$ or $\frac{1}{2}(N-2)$. Except for the change of order, the matrix of coefficients are precisely those indicated and the algorithm employed to solve the tridiagonal system of equations is unchanged.

Owing to the nature of the elements in the matrix A, it is easy to show that the temporary storage required to solve the system of equations comprises, at most,

a one-dimensional array of $\frac{1}{2}(N-1)$ elements. Consequently, the total storage comprises the array storing the (N+1) elements u^{m+1} and temporary vector of $\frac{1}{2}(N-1)$ elements. This compares with Zabusky and Kruskal schemes storage requirement of two arrays each of N+1 elements.

The form of the matrix A will play an important part in efficient computation of the algorithm. We can produce a constant factorization of A that requires only a back substitution at each time level to determine u^{m+1}. Then, we use the Gauss elimination method [46], for solving the tridiagonal system of equations.

3.4-3 The Stability Analysis of The Hopscotch Method

Let us write (3.4.16) for m = m - 1, then

$$u_n^m = u_n^{m-1} - \frac{r}{2}(p+1)(p+2)\left[F_{n+1}^m - F_{n-1}^m\right] - \frac{r\gamma}{2h^2}\left[u_{n+2}^m - u_{n-2}^m\right] + \frac{r\gamma}{h^2}\left[u_{n+1}^m - u_{n-1}^m\right]. \quad (3.4.22)$$

Equations (3.4.22) and (3.4.15), in operator notations, take the forms

$$\left(1+\frac{r\gamma}{2h^2}H_{2x}\right)u_n^m = u_n^{m-1} - \frac{r}{2}(p+1)(p+2)H_x F_n^m + \frac{r\gamma}{h^2}H_x u_n^m, \quad (3.4.23)$$

$$u_n^{m+1} = \left(1-\frac{r\gamma}{2h^2}H_{2x}\right)u_n^m - \frac{r}{2}(p+1)(p+2)H_x F_n^m + \frac{r\gamma}{h^2}H_x u_n^m. \quad (3.4.24)$$

Multiplying (3.4.23) by $\left(1-\frac{r\gamma}{2h^2}H_{2x}\right)$, (3.4.24) by $\left(1+\frac{r\gamma}{2h^2}H_{2x}\right)$ and adding them, to give

$$\left(1+\frac{r\gamma}{2h^2}H_{2x}\right)u_n^{m+1} = \left(1-\frac{r\gamma}{2h^2}H_{2x}\right)u_n^{m-1} - r(p+1)(p+2)H_x F_n^m + \frac{2r\gamma}{h^2}H_x u_n^m. \quad (3.4.25)$$

Hence, the odd-even Hopscotch algorithm is equivalent to the three-level scheme (3.4.25). Notice the difference between it and the Zabusky and Kruskal scheme (3.3.1), which is written in the following operator notations as

$$u_n^{m+1} = u_n^{m-1} - r\,(p+1)(p+2)H_x F_n^m - \frac{r\gamma}{h^2} H_x\, \delta_x^2\, u_n^m. \qquad (3.4.26)$$

Thus, the Hopscotch method has a truncation error $\tau\, o\left(\left(\frac{\tau}{h}\right)^2 + \tau^2 + h^2\right)$ (see Appendix (A)).

Applying the Von-Neumann stability (as in Zabusky and Kruskal scheme) on the Hopscotch scheme (3.4.25) and beginning with Fourier mode (3.4.10), which can written in the form $\left(u_n^{m+1} = g\,(k)\, u_n^m\right)$, the amplification factor g (k) satisfies

$$\left(1+\lambda H_{2x}\right) g\, u_n^m = \left(1-\lambda H_{2x}\right) g^{-1} u_n^m - r(p+1)(p+2)H_x F_n^m + 4\lambda H_x u_n^m, \qquad (3.4.27)$$

where

$$\lambda = \frac{\gamma\, r}{2h^2}.$$

The previous equation gives

$$(1 + 2i\lambda \sin 2\xi)\, g^2 + 2i\, \sin\xi\, (r(p+1)\,(p+2)\, \bar{u}^p - 4\lambda)\, g - (1 - 2i\lambda \sin 2\xi) = 0. \quad (3.4.28)$$

The stability conditions requires that $|\, g| \le 1$, so we are going to study that using Miller's analysis [43] for equation (3.4.28), define

$$f(z) = \left(1 + iA\right) z^2 + iBz - \left(1 - iA\right),$$

where

$$A = 2\lambda \sin 2\xi, \qquad B = 2(r(p+1)\,(p+2)\, \bar{u}^p - 4\lambda) \sin\xi \quad .$$

Suppose $z^* = \bar{z}^{-1}$, $\bar{z}$ is the complex conjugate of z, then

$$f(z^*) = \left(1 + iA\right)\bar{z}^{-2} + iB\bar{z}^{-1} - \left(1 - iA\right)$$

and

$$\overline{f(z^*)} = \left(1 - iA\right) z^{-2} - iBz^{-1} - \left(1 + iA\right).$$

Therefore, with $f^*(z) = z^2\, \overline{f\,(z^*)}$, we have

$$f^*(z) = \left(1 - iA\right) - iBz - \left(1 + iA\right) z^2$$

Then, define the Bezout resultant [43] as

$$\check{f} = [f^*(0)f(z) - f(0)f^*(z)] / z$$

Consequently,

$$f^*(0) = 1 - iA, \text{ and } f(0) = -(1 - iA),$$

then clearly $|f^*(0)| \ngtr |f(0)|$. Now, for f to be Von-Neumann, we must have

$$\check{f} = 0,$$

and

$$f'(z) = 2(1 + iA)z + iB.$$

We require that $|z| \leq 1$ for the function $f'(z)$, that is

$$|z| = \left|\frac{-iB}{2(1+iA)}\right| \leq 1, \qquad \forall \xi.$$

By ensuring $|z| \leq 1$, the resulting zeroes of f (z) lie on the unit circle [43], therefore the scheme is conservative. Now,

$$|z|^2 = \frac{B^2}{4(1+A^2)} \leq 1$$

i. e.

$$\sin^2 \xi \left(r(p+1)(p+2)\bar{u}^p - 4\lambda\right)^2 \leq 1 + 4\lambda^2 \sin^2 2\xi,$$

then

$$\sin^2 \xi \left[r^2 (p+1)^2 (p+2)^2 \bar{u}^{2p} - 8\lambda r (p+1)(p+2)\bar{u}^p + 16\lambda^2 \sin^2 \xi\right] \leq 1.$$

Substituting the value of λ, we have

$$r^2 \sin^2 \xi \left[(p+1)^2 (p+2)^2 \bar{u}^{2p} - \frac{4(p+1)(p+2)\bar{u}^p \gamma}{h^2} + \frac{4\gamma^2}{h^4} \sin^2 \xi\right] \leq 1.$$

The maximum value of the left-hand side of the above equation occurs when $\sin^2 \xi = 1$. Hence, one can deduce

$$r^2 \left[(p+1)(p+2)\bar{u}^p - \frac{2\gamma}{h^2}\right]^2 \leq 1.$$

Then the Hopscotch scheme of the Generalized Kdv equation is conservative and is stable if

$$r\left|(p+1)(p+2)\bar{u}^{p}-\frac{2\gamma}{h^{2}}\right|\leq 1, \tag{3.4.29}$$

where $\bar{u}$ is the maximum value in the range of our investigation.

Notice that again, the stability condition (3.4.29) is agreeing with the corresponding one [41], when p = 1 (Kdv equation) and the factor of nonlinear term in equation (2.3.1) is taken to be 1.

3.5 Numerical results

The studied methods (Zabusky and Kruskal (Z&K) and Hopscotch) for the solution of the Generalized Kdv equation are written with Fortran programs and developed at some different values of the parameter p (a list of them is given in Appendix (c)). They deal with a one soliton solution.

Generally, the accuracy between the separated numerical solutions of the two methods and the analytical one is estimated by calculating L_∞, i.e. it is done by computing the maximum difference between them at each time (finite time). The solitary wave solution (2.3.47) for the Generalized Kdv (2.3.1) is written as

$$u(x,t) = \left(\frac{v}{2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}(A_1 x - B_1 t + D), \qquad (3.5.1)$$

where $A_1 = \frac{p}{2}\left(\frac{v}{\gamma}\right)^{\frac{1}{2}}$, $B_1 = \frac{pv}{2}\left(\frac{v}{\gamma}\right)^{\frac{1}{2}}$ and $D = \frac{p\Delta}{2}$.

The initial condition is given by

$$u(x,0) = \left(\frac{v}{2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}(A_1 x + D), \qquad (3.5.2)$$

and the periodic boundary condition is

$$u(0,t) = u(2,t), \qquad (3.5.3)$$

for all t and $0 \le x \le 2$.

Greig and Morris [8] used the following initial values of the required parameters to obtain the numerical solution of the Kdv equation

$$v = 0.3, \qquad \Delta = -6 \qquad \text{and } \gamma = 4.84 \times 10^{-4}. \qquad (3.5.4)$$

Upon starting with the same parameters and the number of mesh points are taken 200, i.e. $h = 0.01$ with fixed $\Delta t = 0.0005$. The two methods are compared in table (1) by calculating L_∞ error where p equal to 1,2 and 3.

Table (1) Computing L_∞ beginning at T = 0.0 and ending at T = 4.0 with v = 0.3 and $\Delta t = 0.0005$

Value of p	Mesh size		Time	$\lVert u_{num} - u_{anal} \rVert$	
	Δt^*	Δx^*		Z&K method	Hopscotch method
1 (Kdv) equation	5×10^{-4}	0.01	0.25	0.001168	0.001217
			0.50	0.001120	0.000942
			0.75	0.001296	0.001113
			1.0	0.001601	0.001442
			2.0	0.003377	0.002309
			3.0	0.003728	0.002549
			4.0	0.005588	0.002549
2 (Mkdv) equation	5×10^{-4}	0.01	0.25	0.006145	0.001542
			0.50	0.010728	0.001700
			0.75	0.016745	0.002019
			1.0	0.021883	0.003415
			2.0	0.042984	0.003945
			3.0	0.063079	0.007118
			4.0	0.083683	0.010589
3	5×10^{-4}	0.01	0.25	0.027798	0.013668
			0.50	0.052877	0.027456
			0.75	0.080994	0.040975
			1.0	0.108180	0.057049
			2.0	0.211296	0.119694
			3.0	0.298453	0.177837
			4.0	0.366519	0.238246

$^*\Delta x$ = the increment in x

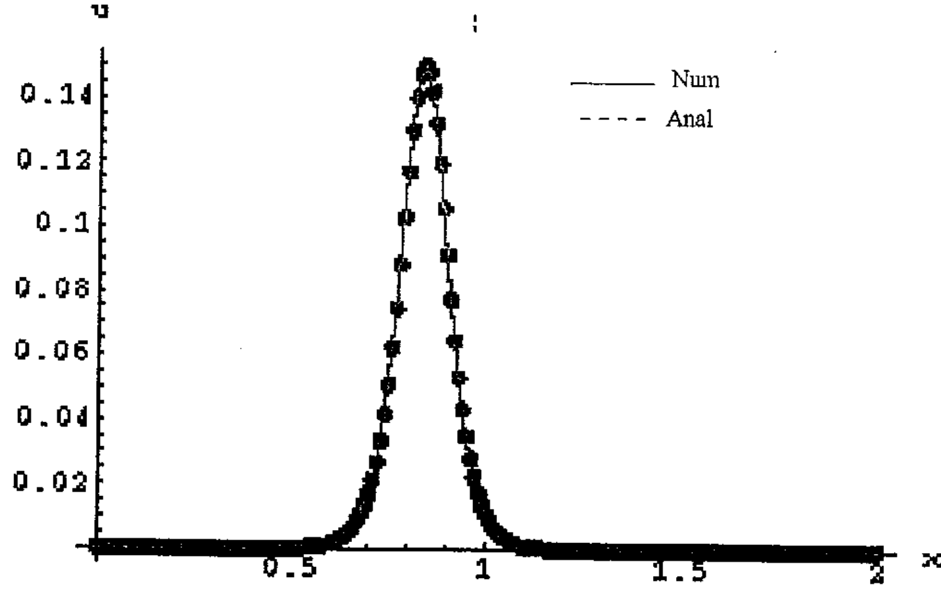
$^*\Delta t$ = the increment in t

$^*L_\infty$ error = $\lVert u_{num} - u_{anal} \rVert$, u_{num} is the numerical solution and u_{anal} is the exact solution at the grid point $(n\Delta x, m\Delta t)$ for all n , m.

Although, the maximum difference between the numerical solution of the two methods and the analytical one is increasing with time in each case of p and also for increasing p at the same time in table (1), the Hopscotch method has almost less error than the method of Zabusky and Kruskal.

Also, Figs[3.1]and[3.2]represent the numerical solutions for p = 1 (the Kdv equation) of the two methods at t = 2,4 as illustrated examples with the analytical one at separating plots. They generally coincide with the analytical one where the maximum difference is less than 6×10^{-3} for Zabusky and Kruskal method and 3×10^{-3} for the Hopscotch method.

(a) Z &K



(b) Hopscotch

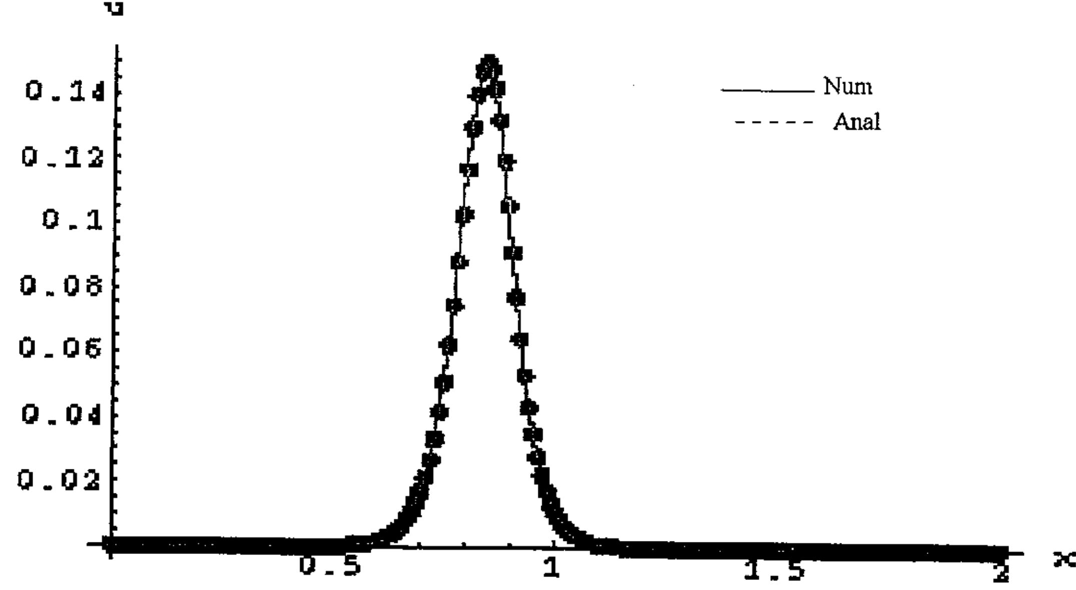
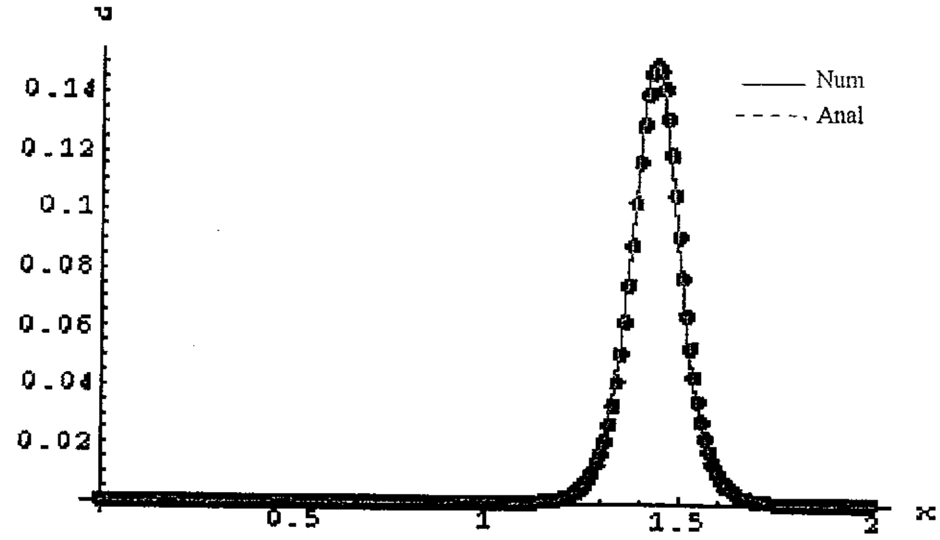


Fig (3.1) represents the numerical and analytical solutions for $p = 1$ (Kdv) and $t = 2$ when $v = 0.3$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K



(b) Hopscotch

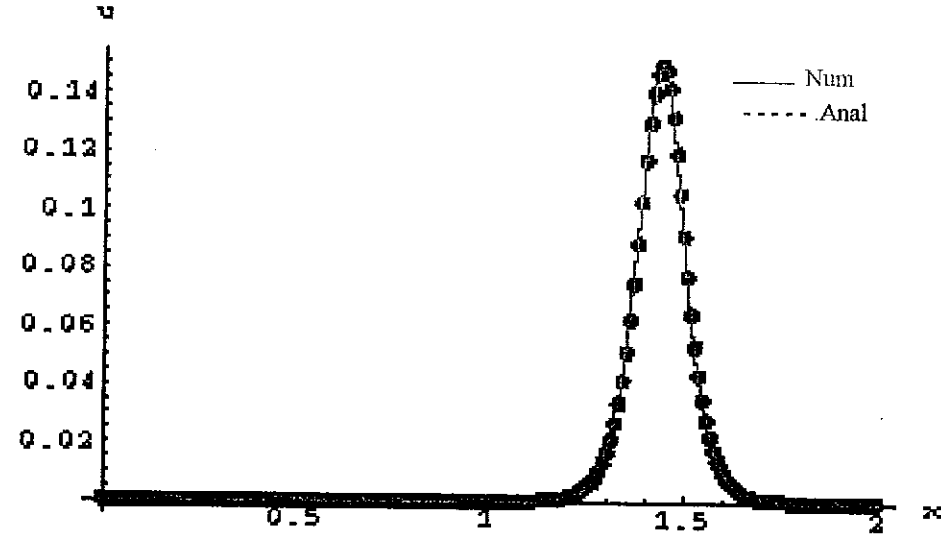
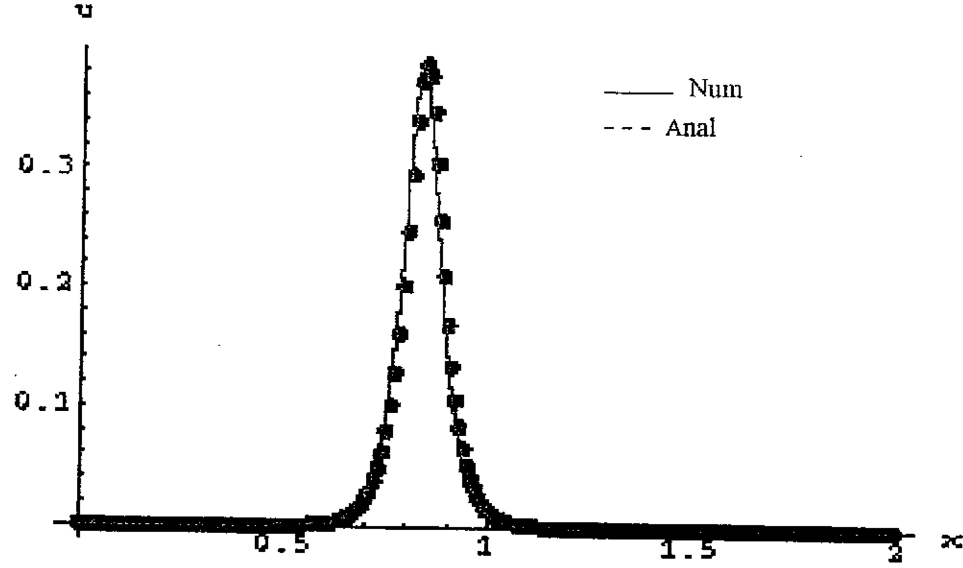


Fig (3.2) represents the numerical and analytical solutions for p =1 (Kdv) and t = 4 when v = 0.3 , $\Delta t = 5\times10^{-4}$.

The numerical solutions of the Mkdv equation (p = 2) are compared by a similar manner with the analytical one as shown in Figs [3.3] and [3.4]. The Hopscotch method is still the better where L_∞ error is less than 2×10^{-2} at the calculated times against 9×10^{-2} for Zabusky and Kruskal method.

In the case p = 3, the numerical and the analytical solutions are shown in Figs [3.5] and [3.6]. The value of L_∞ (L_∞ <0.4 for Z&K, L_∞ < 0.3 for Hopscotch) is greater than the previous cases at p = 1 and p = 2 but the Hopscotch method has less errors. Also, the numerical solution seemed to be behind the analytical at advancing time for Zabusky and Kruskal method but the Hopscotch numerical results are in the front of it.

(a) Z &K



(b) **Hopscotch**

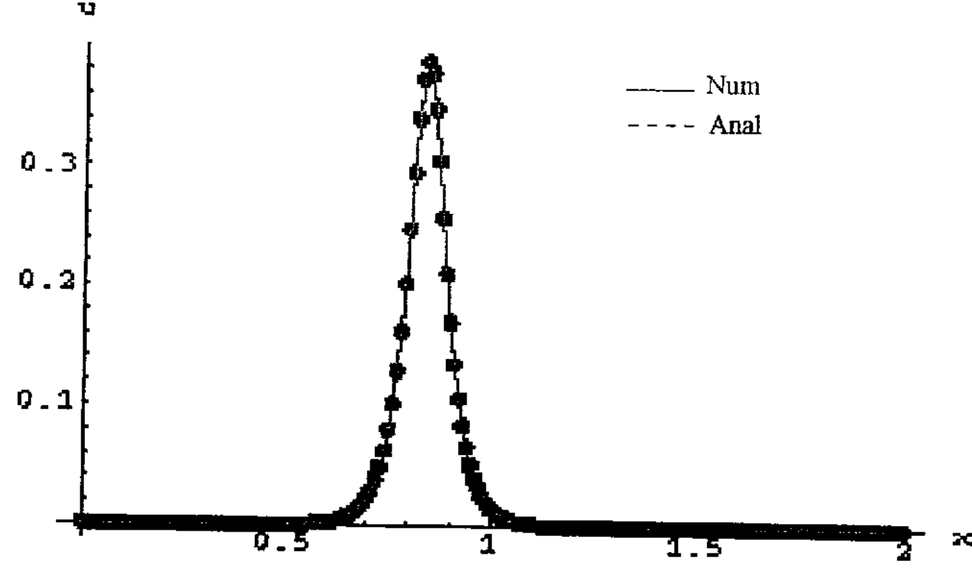
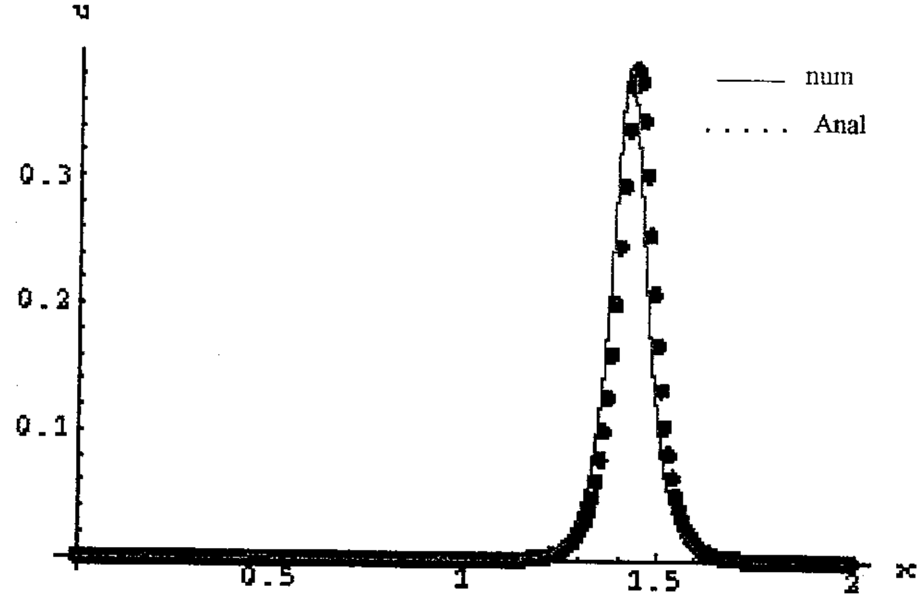


Fig (3.3) represents the analytical and numerical solutions for p =2 (Mkdv) and $t = 2$ when $v = 0.3$, $\Delta t = 5\times10^{-4}$.

(a) Z &K



(b) Hopscotch

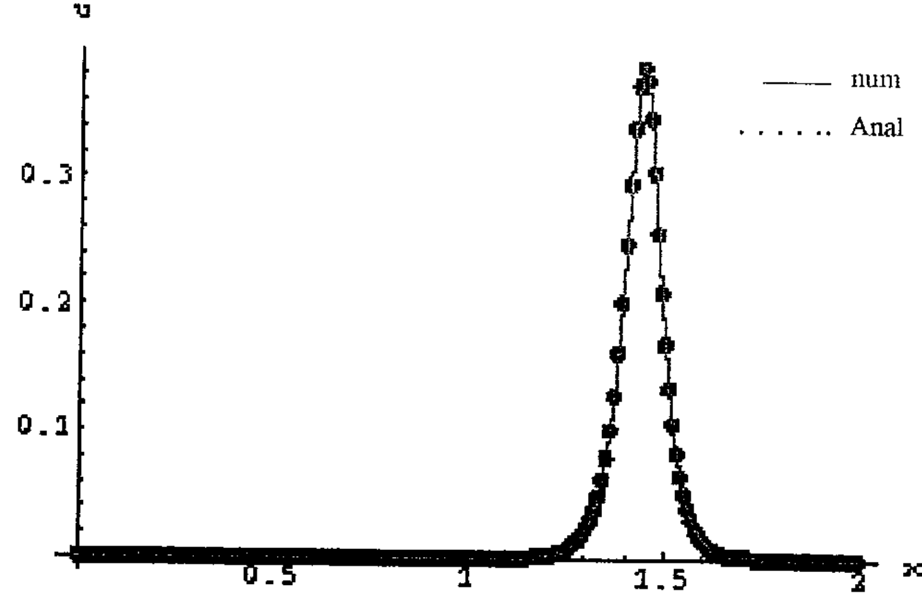
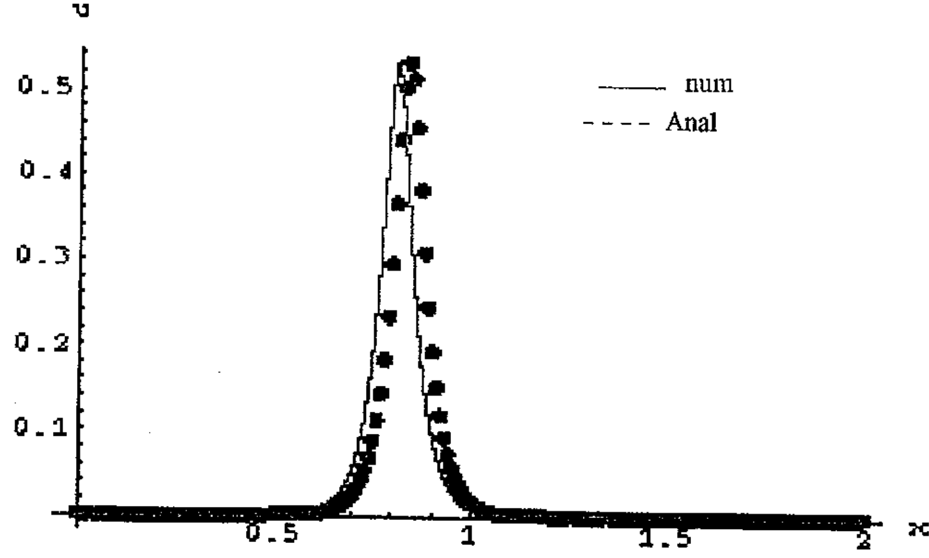


Fig (3.4) represents the analytical and numerical solutions for p = 2 (Mkdv) and t =4 when $v = 0.3$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K



(b) Hopscotch

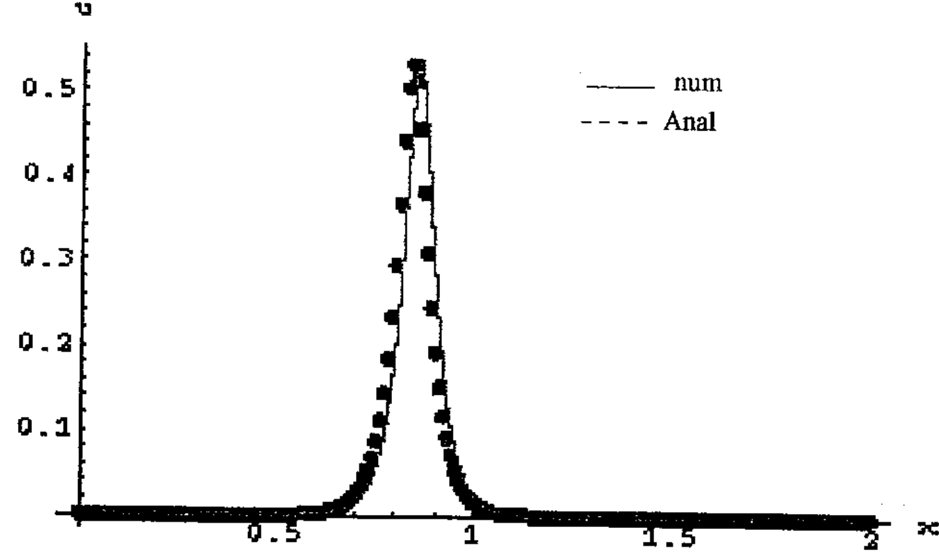
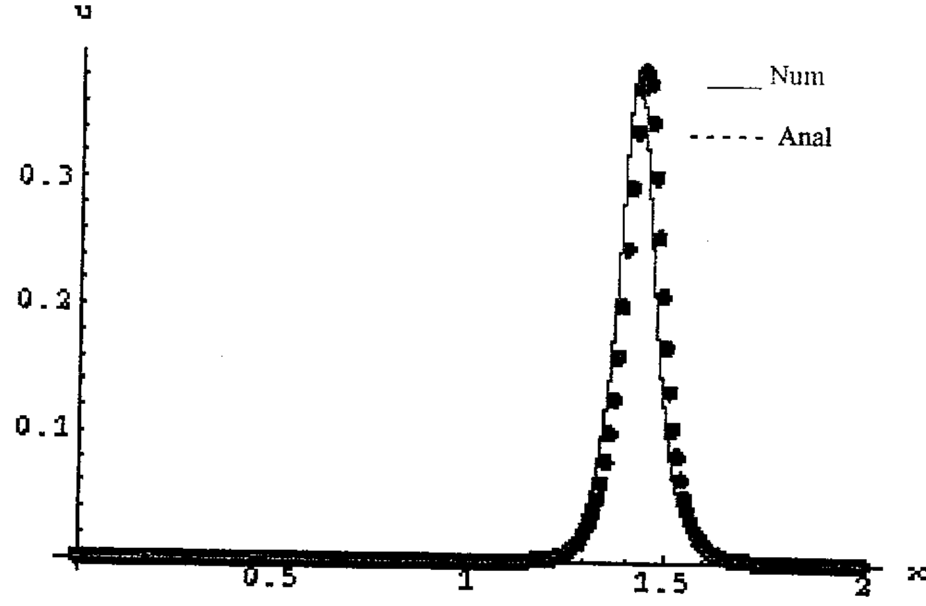


Fig (3.5) represents the analytical and numerical solutions for $p = 3$ and $t = 2$ when $v = 0.3$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K



(b) Hopscotch

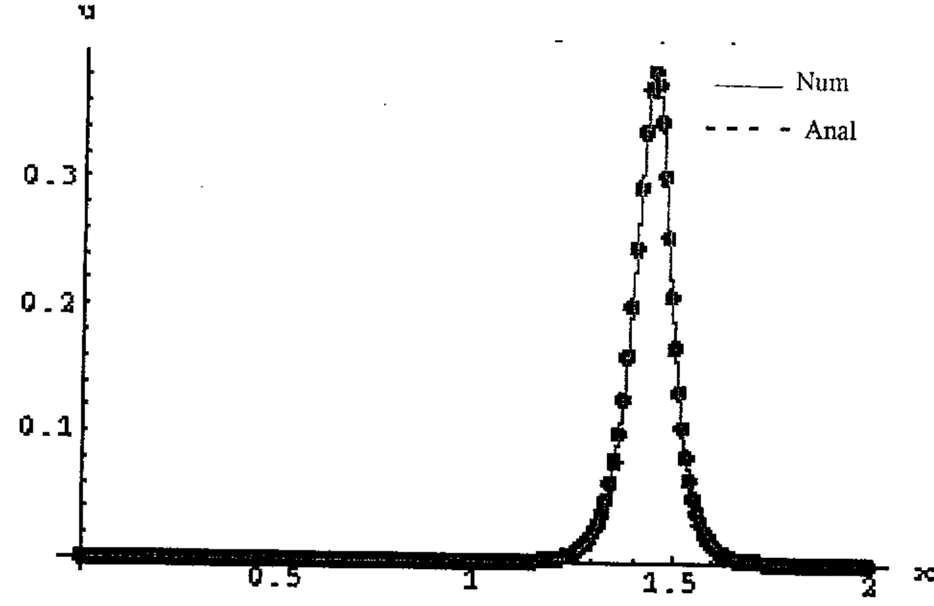


Fig (3.6) represents the analytical and numerical solutions for $p = 3$ and $t = 4$ when $v = 0.3$, $\Delta t = 5\times10^{-4}$.

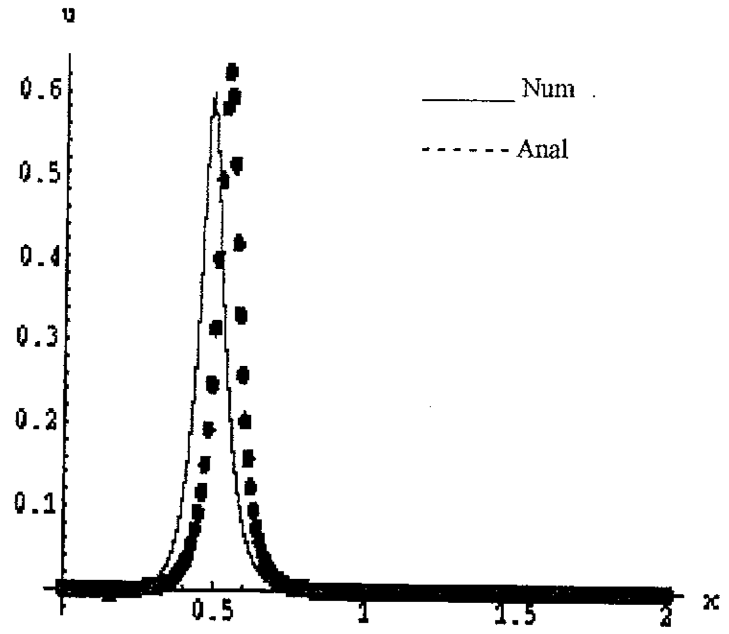
For increasing the parameter p the value of L_∞ is increasing. Table (2) represents the values of L_∞ for Zabusky & Kruskal method at p = 4, 5, 6 and some results of it for Hopscotch method at p = 4 only. The stability condition (3.4.31) of Hopscotch scheme is not satisfied at p = 5, 6 and at increasing time for p = 4. Although, the stability condition (3.3.16) of Zabusky and Kruskal scheme is satisfied, the numerical results are generally unstable.

Table (2) Computing L_∞ error beginning at T = 0.0 and ending at T = 4.0 with v = 0.3 and $\Delta t = 0.0005$

Value of p	Mesh size		Time	$\|u_{num} - u_{anal}\|$	
	Δt^*	Δx^*		Z&K method	Hopscotch method
4	5×10^{-4}	0.01	0.25	0.082713	0.072102
			0.50	0.181944	0.194612
			0.75	0.28745	0.360915
			1.0	0.378586	0.524049
			2.0	0.574458	Unstable
			3.0	0.616982	Unstable
			4.0	0.615970	Unstable
5	5×10^{-4}	0.01	0.25	0.217564	Unstable
			0.50	0.447458	Unstable
			0.75	0.575376	Unstable
			1.0	0.644106	Unstable
			2.0	0.699315	Unstable
			3.0	0.656860	Unstable
			4.0	0.566426	Unstable
6	5×10^{-4}	0.01	0.25	0.420576	Unstable
			0.50	0.606898	Unstable
			0.75	0.674808	Unstable
			1.0	0.720267	Unstable
			2.0	0.726131	Unstable
			3.0	0.741968	Unstable
			4.0	0.765306	Unstable

Some figures illustrate the disturbance between the numerical results and the analytical one as follows. Fig [3.7] shows the difference between them at t = 1.0 for p = 4. While Fig [3.8], compares the numerical solution of Zabusky and Kruskal only with the analytical one at t = 3 for p = 4. Also Figs [3.9],[3.10],[3.11] and [3.12] show the disturbance between the numerical results of Zabusky and Kruskal method at t = 1 and 3 for each separation case of p = 5 and 6.

(a) Z &K



(b) Hopscotch

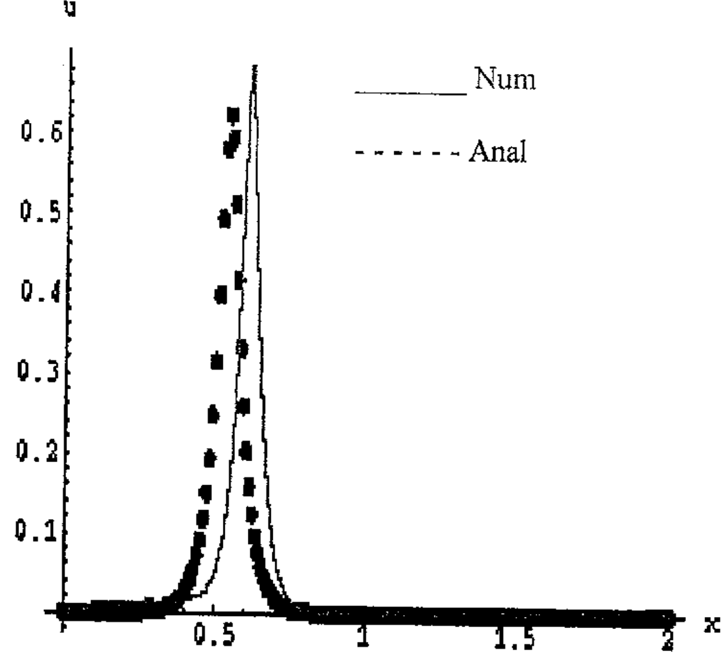


Fig (3.7) represents the analytical and numerical solutions for $p = 4$, and $t = 1$ when $v = 0.3$, $\Delta t = 5\times10^{-4}$.

(a) Z &K

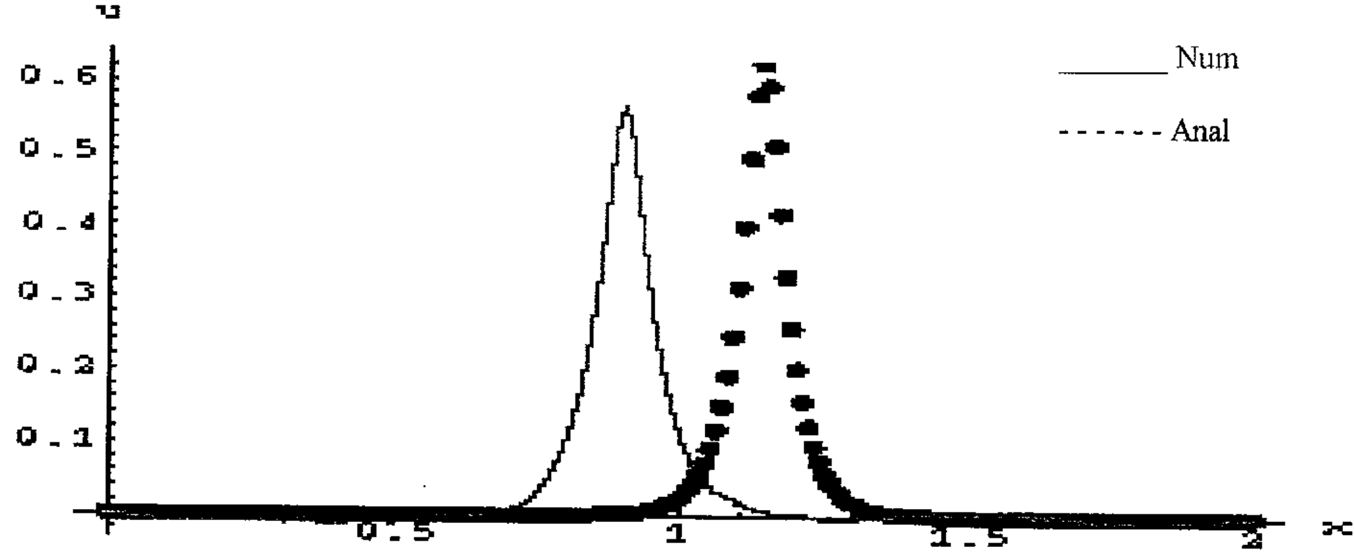


Fig (3.8) represents the analytical and numerical solutions for $p = 4$, and $t = 3$ when $v = 0.3$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K

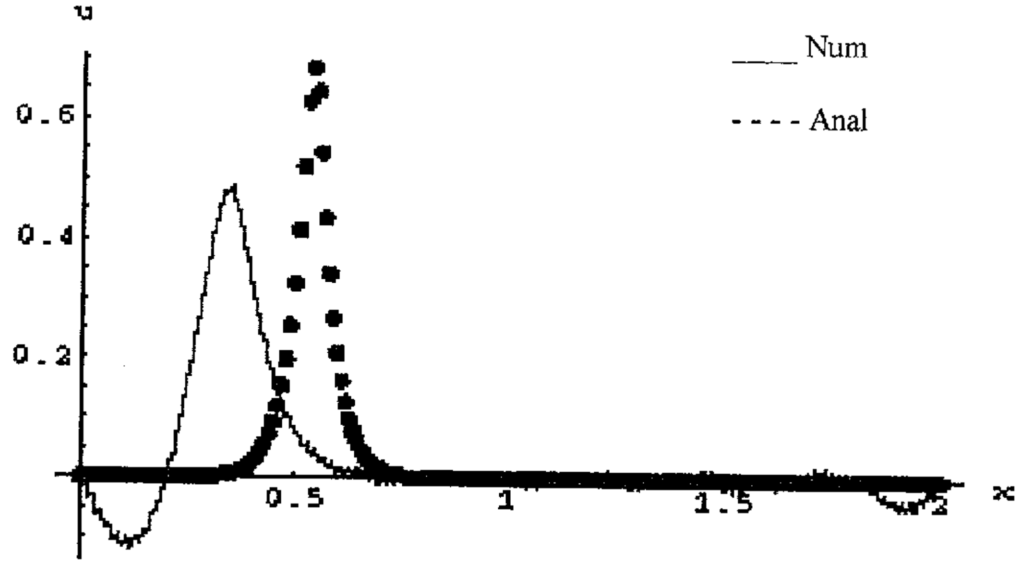


Fig (3.9) represents the analytical and numerical solutions for p = 5, and t =1 when v = 0.3, $\Delta t = 5\times10^{-4}$.

(a) Z &K

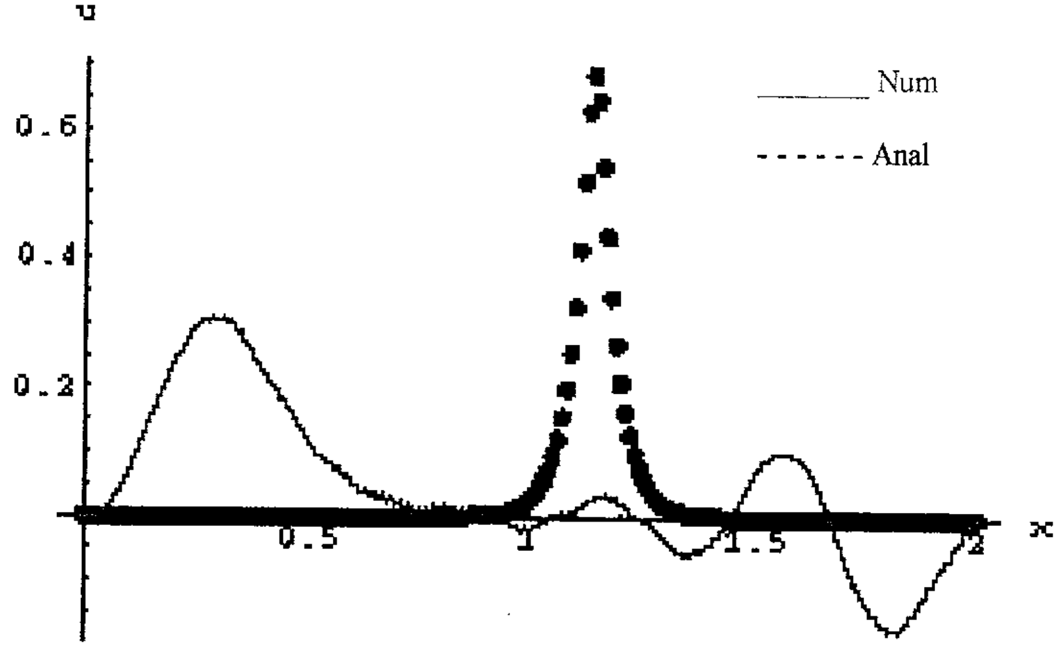


Fig (3.10) represents the analytical and numerical solutions for p = 5, and t =3 when v = 0.3, $\Delta t = 5\times10^{-4}$.

(a) Z &K

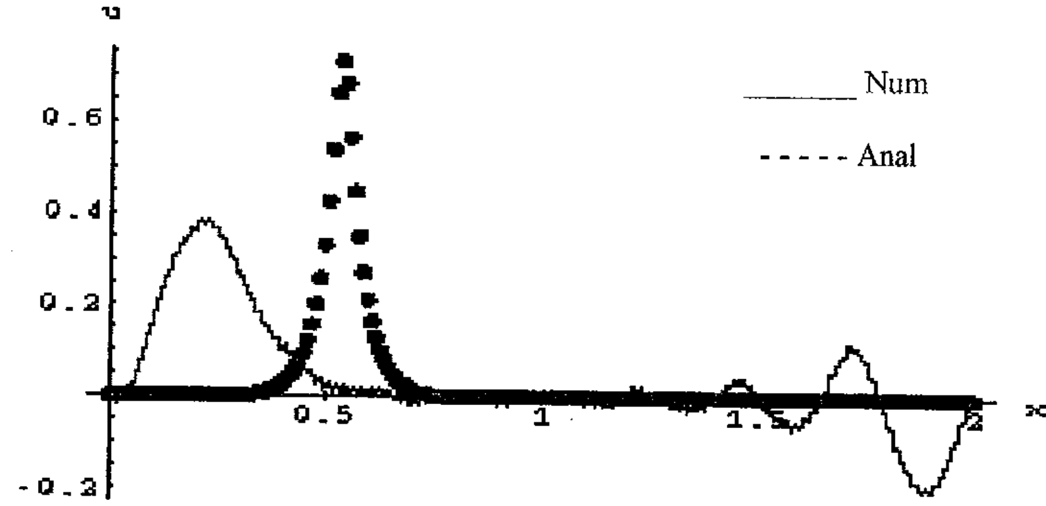


Fig (3.11) represents the analytical and numerical solution for p = 6, and t =1, when $v = 0.3$, $\Delta t = 5\times10^{-4}$.

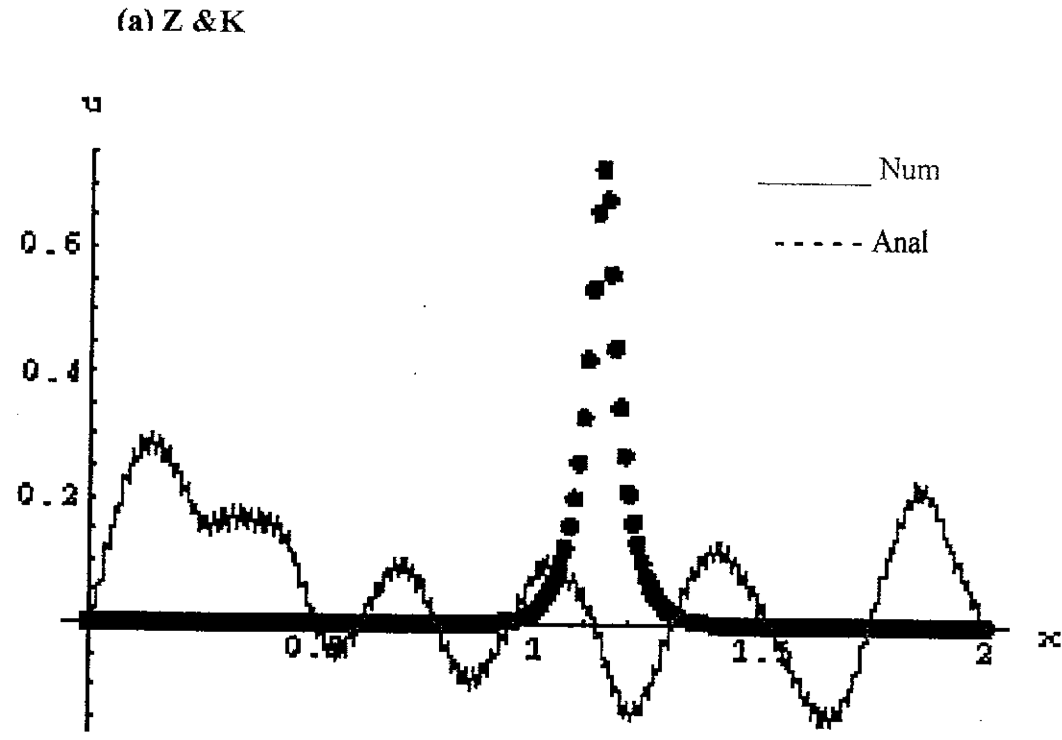


Fig (3.12) represents the analytical and numerical solutions for $p = 6$, and $t = 3$ when $v = 0.3$, $\Delta t = 5 \times 10^{-4}$.

On improving the numerical results for p = 4, 5, 6, the value of Δt is decreasing especially to satisfy the stability condition (3.4.31) of Hopscotch method. In this direction, firstly, we choose $\Delta t = 1\times10^{-4}$ but the results are not improved and the numerical solutions of the two methods are take the same forms as Figs ([3.7] and [3.8]) at p = 4. For p = 5 and 6 the results are greatly similar as the previous case as indicated in table (2) and Figs ([3.9]-[3.12]). The values of L_∞ in table (3) are showing those result when $\Delta t = 1\times10^{-4}$.

Table (3) Computing L_∞ beginning at T = 0.0 and ending at T = 4.0 with v = .3 and $\Delta t = 1\times10^{-4}$

Value of p	Mesh size		Time	L_∞^*	
	Δt^*	Δx^*		Z&K method	Hopscotch method
4	1×10^{-4}	0.01	0.25	0.083624	0.071794
			0.50	0.182838	0.195914
			0.75	0.287177	0.362127
			1.0	0.378297	0.526798
			2.0	0.575261	Unstable
			3.0	0.616055	Unstable
			4.0	0.615830	Unstable
5	1×10^{4}	0.01	0.25	0.217877	Unstable
			0.50	0.447446	Unstable
			0.75	0.575658	Unstable
			1.0	0.645437	Unstable
			2.0	0.700406	Unstable
			3.0	0.658482	Unstable
			4.0	0.565172	Unstable
6	1×10^{-4}	0.01	0.25	0.422020	Unstable
			0.50	0.606572	Unstable
			0.75	0.674073	Unstable
			1.0	0.718169	Unstable
			2.0	0.726447	Unstable
			3.0	0.740562	Unstable
			4.0	0.765414	Unstable

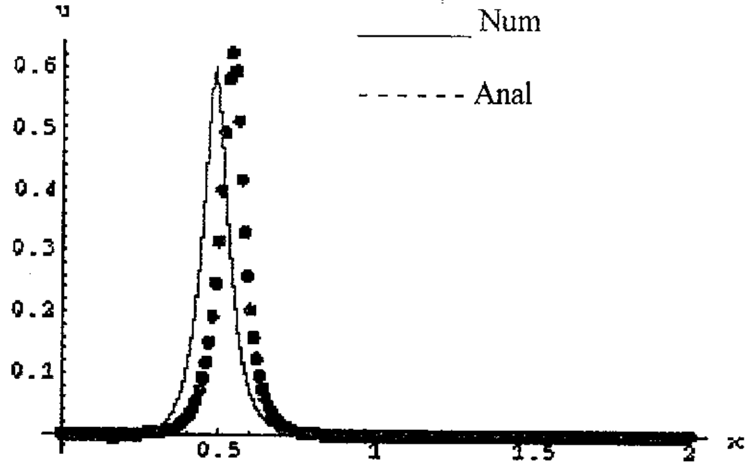
In another trail to overcome this difficulty and get rid these numerical drawback, we will take $\Delta t = 4\times10^{-5}$. Although, the stability condition of the two methods are satisfied for p = 4 and 5 when Δt is taken equal to 4×10^{-5}, the difference between the numerical solutions and the analytical one is still large as the time increases as indicated in table (4), also, the Hopscotch stability condition (3.4.31) is not satisfied at p = 6.

Table (4) Computing L_∞ beginning at T = 0.0 and ending at T = 4.0 with v = 0.3 and $\Delta t = 4 \times 10^{-5}$

Value of p	Mesh size		Time	L_∞^*	
	Δt^*	Δx^*		**Z&K method**	**Hopscotch method**
4	4×10^{-5}	0.01	0.25	0.083727	0.071937
			0.50	0.182921	0.196502
			0.75	0.286948	0.363418
			1.0	0.377782	0.528008
			2.0	0.574227	0.748209
			3.0	0.614400	0.664682
			4.0	0.613547	0.694598
5	4×10^{-5}	0.01	0.25	0.218002	0.936451
			0.50	0.447304	0.716740
			0.75	0.575528	0.687896
			1.0	0.645357	0.796676
			2.0	0.699567	0.780035
			3.0	0.657198	0.804888
			4.0	0.562790	0.785805
6	4×10^{-5}	0.01	0.25	0.422002	Unstable
			0.50	0.606482	Unstable
			0.75	0.673943	Unstable
			1.0	0.718277	Unstable
			2.0	0.726498	Unstable
			3.0	0.749524	Unstable
			4.0	0.763902	Unstable

Some figures are reported to demonstrate the disturbance between the numerical and analytical results as follows. Figs ([3.13]-[3.16]) show the difference between the numerical solutions for each method at t = 1.0 and 3.0 for p = 4 and 5, while Figs [3.17] compares the numerical solutions of Zabusky & Kruskal only with the analytical one at $t = 1.0$ and 3.0 for $p = 6$.

(a) Z &K



(b) Hopscotch

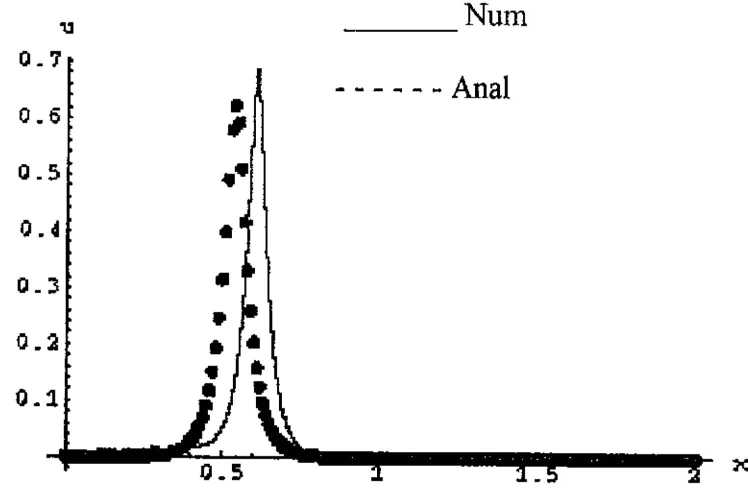
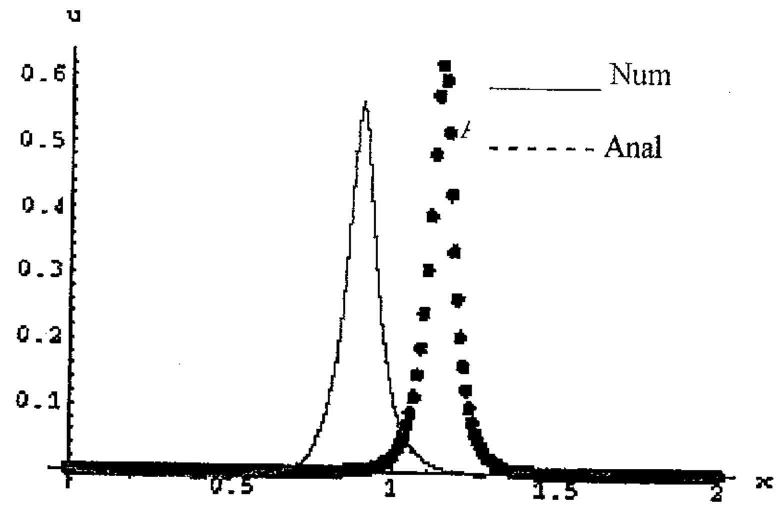


Fig (3.13) represents the analytical and numerical solutions for $p = 4$, and $t = 1$ when $v = 0.3$, $\Delta t = 4 \times 10^{-5}$.

(a) Z &K



(b) Hopscotch

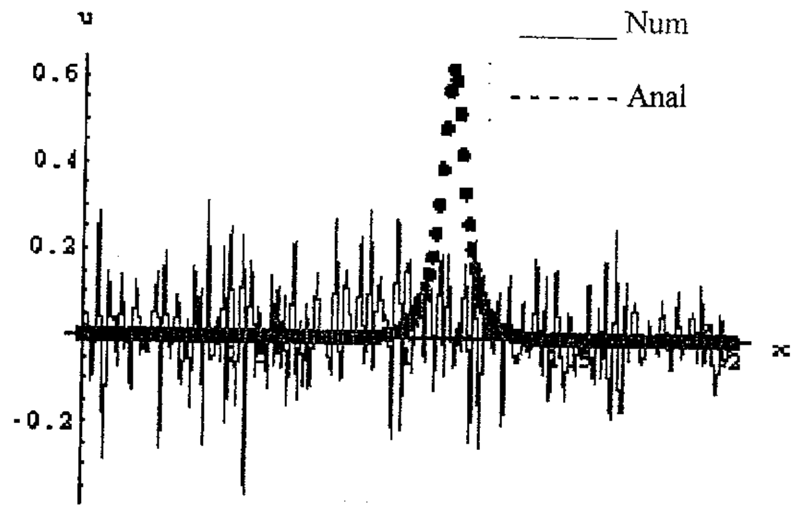
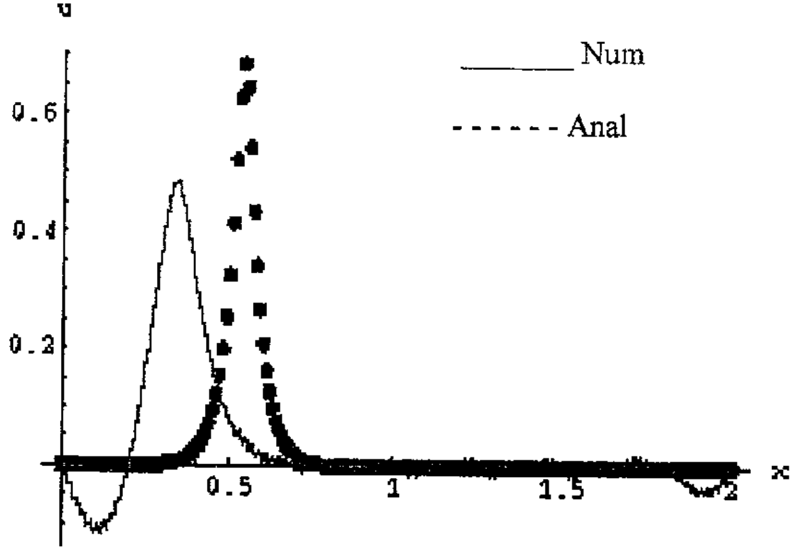


Fig (3.14) represents the analytical and numerical solutions for $p = 4$, and $t = 3$ when $v = 0.3$, $\Delta t = 4 \times 10^{-5}$.

(a) Z &K



(b) Hopscotch

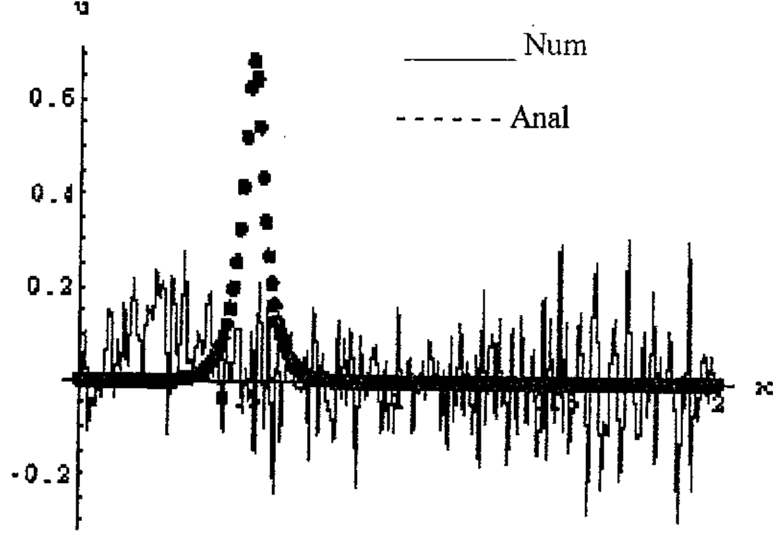
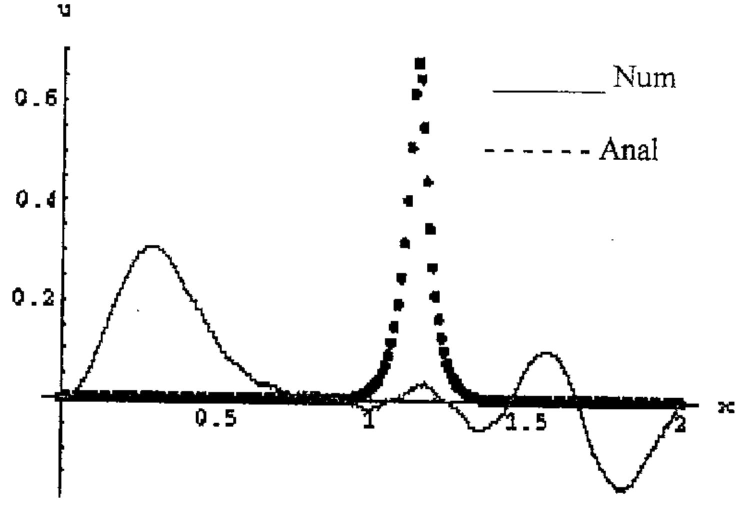


Fig (3.15) represents the analytical and numerical solutions for $p = 5$, and $t = 1$ when $v = 0.3$, $\Delta t = 4 \times 10^{-5}$.

(a) Z &K



(b) Hopscotch

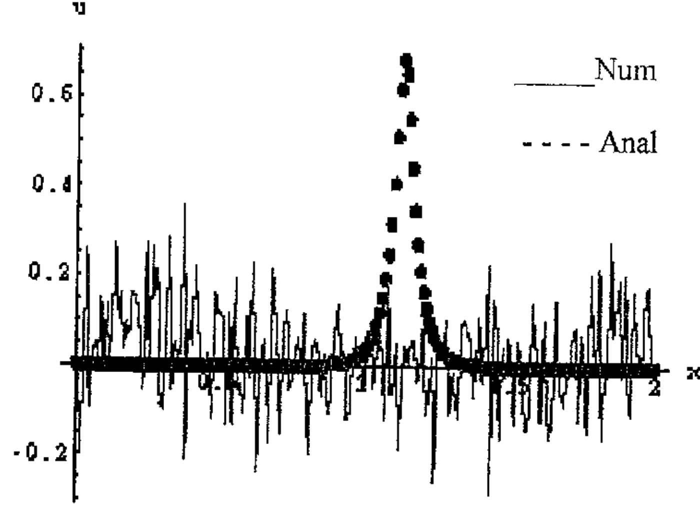
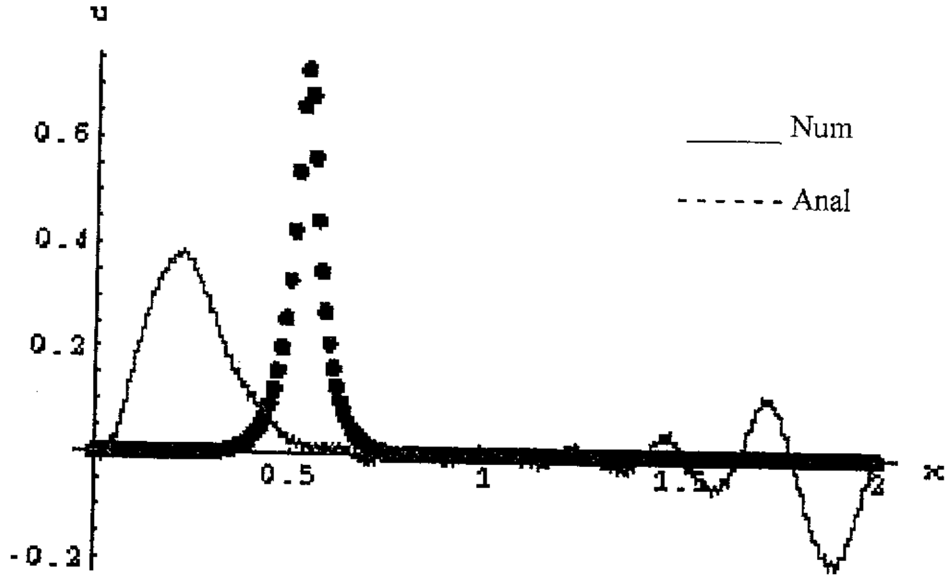


Fig (3.16) represents the analytical and numerical solutions for p = 5, and t =3 when v = 0.3, $\Delta t = 4\times10^{-5}$.

(a) Z &K



(a) Z &K

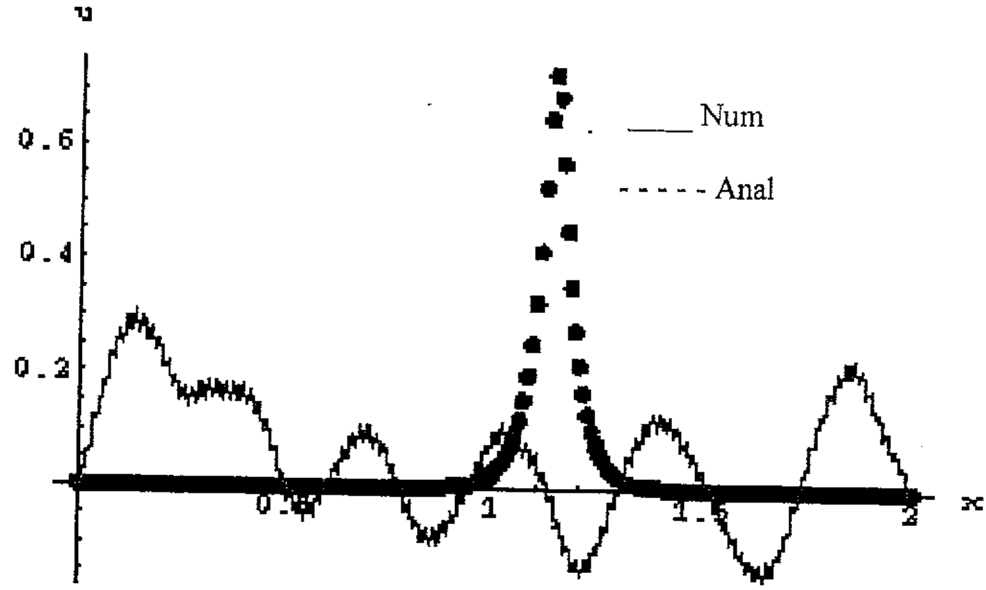


Fig (3.17) represents the analytical and numerical solutions for p = 6, when t =1 and 3, v = 0.3 and $\Delta t = 4\times10^{-5}$.

Notice that, when we take $\Delta t = 2\times10^{-5}$ the Hopscotch stability condition (3.4.31) for p = 5,6 satisfies, but the numerical solutions are still disturbance and L_∞ error are calculated as indicated in table (5).

Table (5) Computing L_∞ beginning at T = 0.0 and ending at T = 4.0 with v = 0.3 and $\Delta t = 2\times10^{-5}$

Value of p	Mesh size		Time	L_∞^*	
	Δt^*	Δx^*		Z&K method	Hopscotch method
5	2×10^{-5}	0.01	0.25	0.218021	0.451579
			0.50	0.447264	0.694245
			0.75	0.575995	0.797532
			1.0	0.645059	0.786561
			2.0	0.699157	0.694459
			3.0	0.657024	0.761766
			4.0	0.562106	0.735630
6	2×10^{-5}	0.01	0.25	0.421980	0.706700
			0.50	0.606414	0.785085
			0.75	0.673095	0.745878
			1.0	0.717944	0.804685
			2.0	0.726154	0.723388
			3.0	0.751588	0.876807
			4.0	0.762822	0.779924

Also, we would mentioned that for small values of Δt then the previous stability condition is satisfied without improving the numerical results with higher values of the parameter p especially when the time increases. The previous analysis motivates to change the velocity v, depending on the amplitude of the solitary, which is increasing when the parameter p increases.

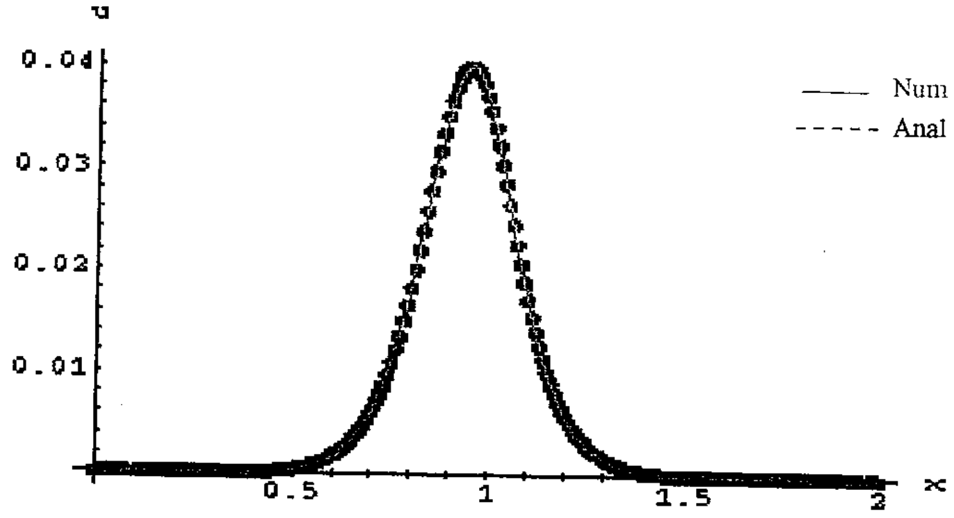
On choosing v = 0.08, table (6) represents the value of L_∞ at $\Delta x = 0.01$ and $\Delta t = 5\times10^{-4}$ for p = 1, 2...6. Generally, we need to increase time than the previous case for comparing the solitary waves at the same x. Some unstable results appears for p =5, 6 with the Hopscotch method when time increases.

Table (6) Computing L_∞ beginning at T = 0.0 and ending at T = 6.0 with v = 0.08 and $\Delta t = 5\times10^{-4}$

Value of P	Mesh size		Time	L_∞^*	
	Δt^*	Δx^*		Z&K method	Hopscotch method
1 (Kdv) equation	5×10^{-4}	0.01	0.25	0.000434	0.000437
			0.50	0.000417	0.000454
			0.75	0.000416	0.000420
			1.0	0.000409	0.000349
			2.0	0.000326	0.000362
			4.0	0.000394	0.000355
			6.0	0.000355	0.000324
2 (Mkdv) equation	5×10^{-4}	0.01	0.25	0.001099	0.001104
			0.50	0.001135	0.001143
			0.75	0.001082	0.001036
			1.0	0.000993	0.000874
			2.0	0.001011	0.000854
			4.0	0.001995	0.000879
			6.0	0.002311	0.001291
3	5×10^{-4}	0.01	0.25	0.001555	0.001530
			0.50	0.001647	0.001734
			0.75	0.002021	0.001414
			1.0	0.002699	0.001734
			2.0	0.005174	0.003207
			4.0	0.010359	0.005762
			6.0	0.010228	0.009360
4	5×10^{-4}	0.01	0.25	0.003260	0.002360
			0.50	0.009868	0.003277
			0.75	0.007078	0.004851
			1.0	0.009201	0.006463
			2.0	0.020477	0.015183
			4.0	0.050972	0.041694
			6.0	0.090080	0.080575
5	5×10^{-4}	0.01	0.25	0.006738	0.005014
			0.50	0.012533	0.009615
			0.75	0.019894	0.015904
			1.0	0.028350	0.024268
			2.0	0.079858	0.089744
			4.0	0.250492	Unstable
			6.0	0.399623	Unstable
6	5×10^{-4}	0.01	0.25	0.013470	0.010815
			0.50	0.030817	0.027527
			0.75	0.054730	0.058404
			1.0	0.083981	0.121175
			2.0	0.253378	Unstable
			4.0	0.472922	Unstable
			6.0	0.553725	Unstable

Some results are plotted for each method for p = 1, 2, 3, 4, 5, 6 at t = 6 as in Figs ([3.18]-[3.25]).

(a) Z &K



(b) Hopscotch

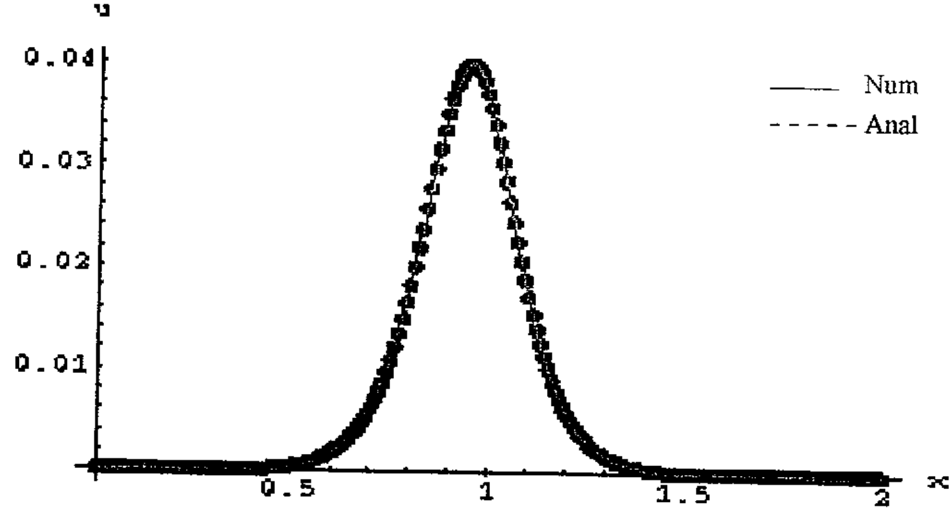
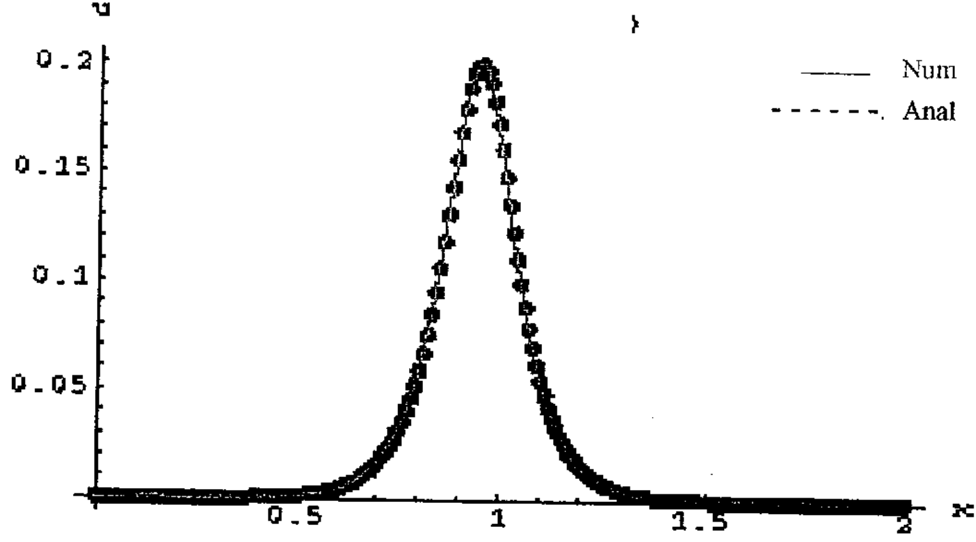


Fig (3.18) represens the analytical and numerical solution for p =1 , and t = 6 when $v = 0.08$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K



(b) Hopscotch

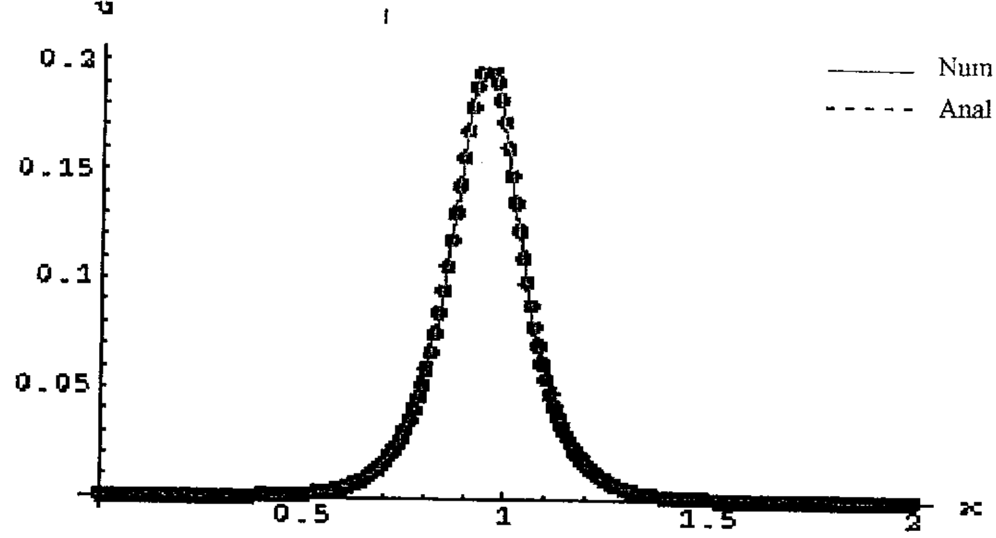
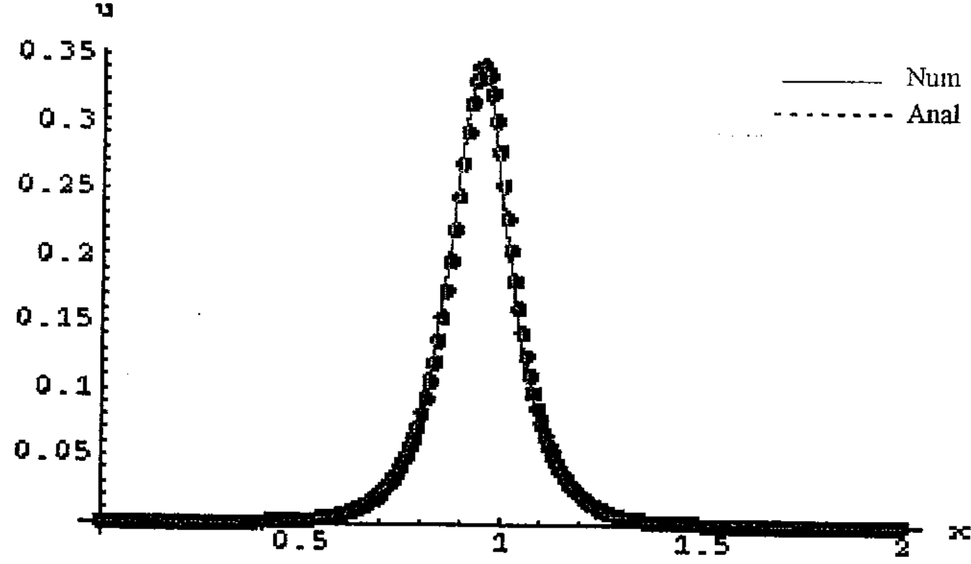


Fig (3.19) represents the analytical and numerical solutions for $p = 2$, and $t = 6$ when $v = 0.08$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K



(b) Hopscotch

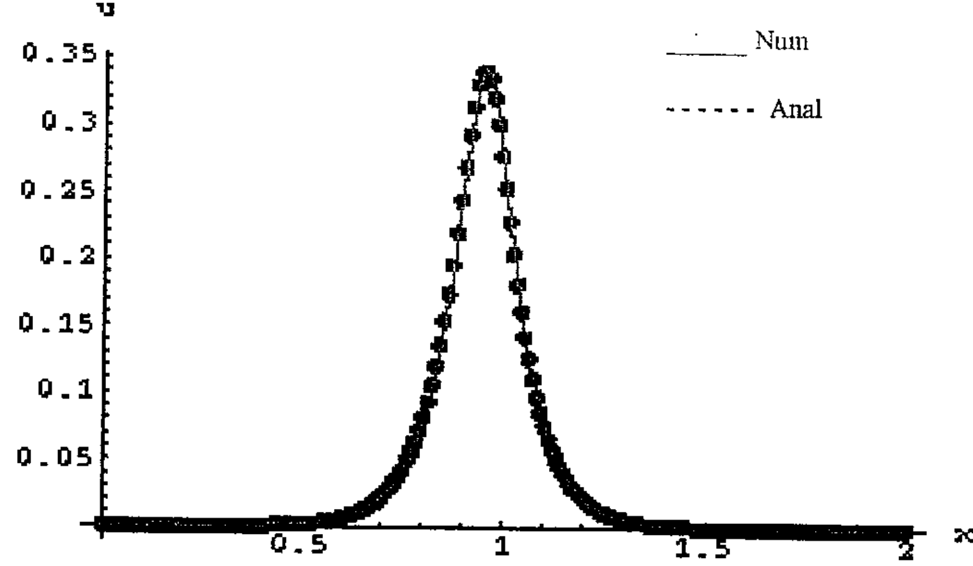
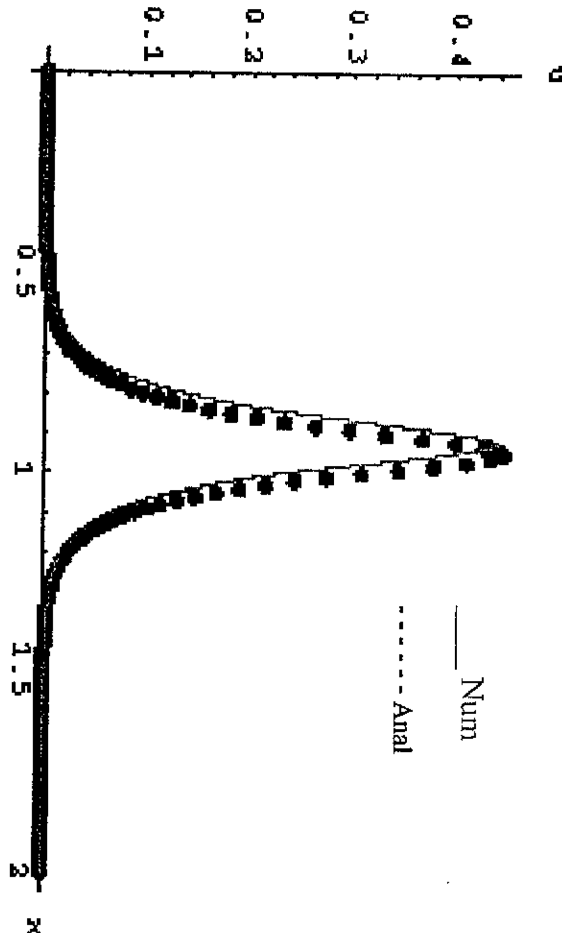


Fig (3.20) represens the analytical and numerical solutions for $p = 3$, and $t = 6$ when $v = 0.08$, $\Delta t = 5 \times 10^{-4}$



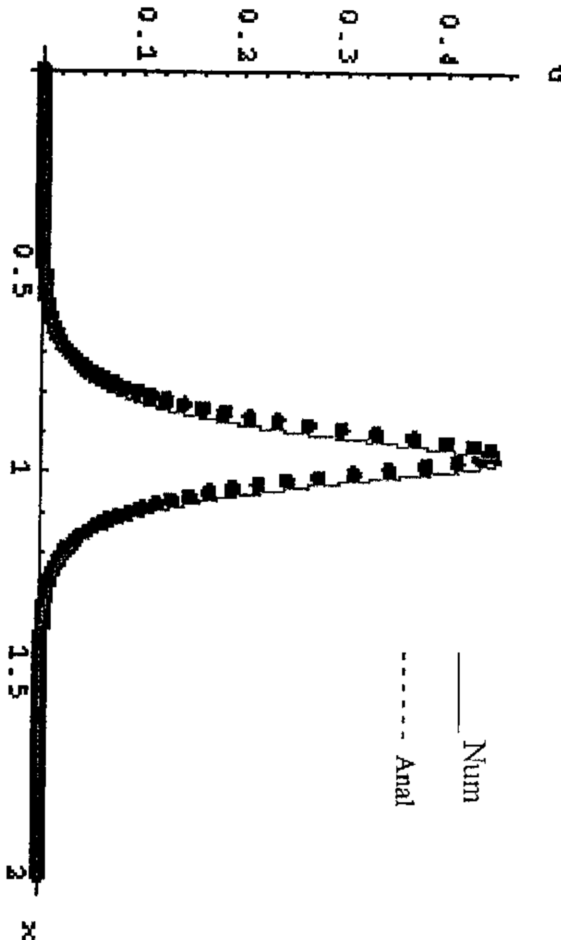
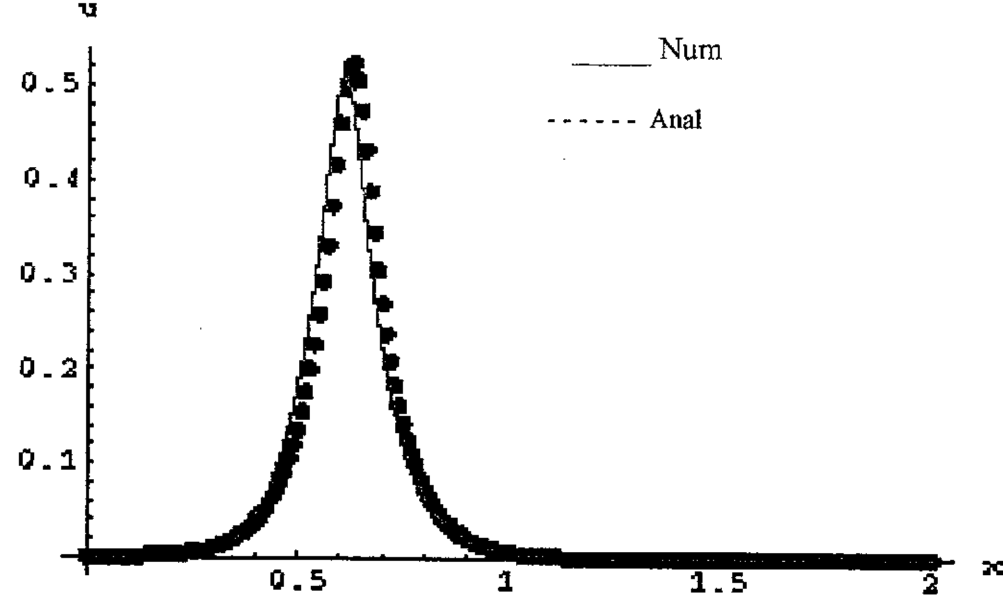


Fig (3.21) represens the analytical and numerical solutions for $p = 4$, and $t = 6$ when $v = 0.08$, $\Delta t = 5\times10^{-4}$.

(a) Z &K



(b) Hopscotch

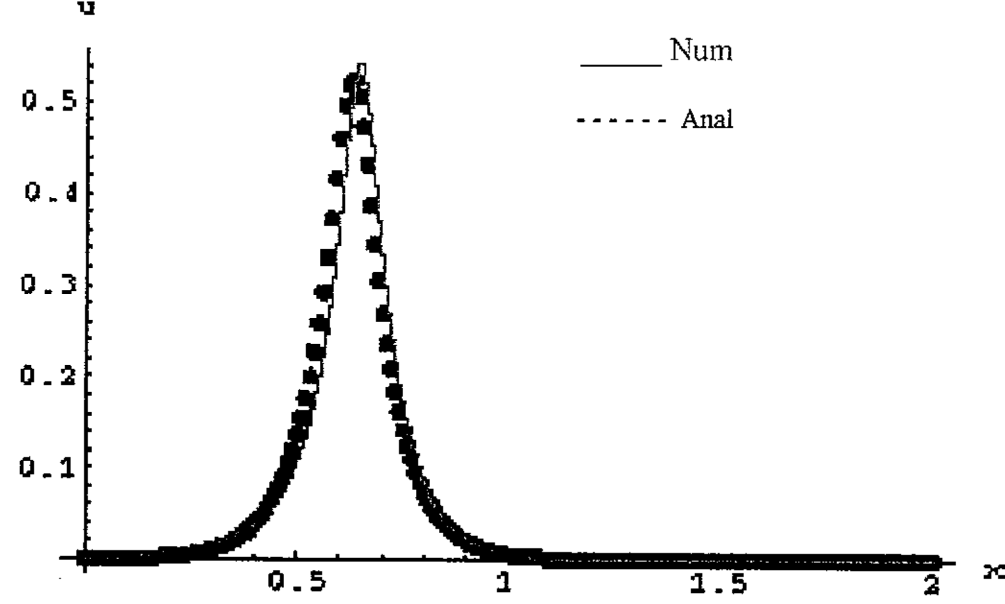


Fig (3.22) represents the analytical and numerical solutions for $p = 5$, and $t = 2$ when $v = 0.08$, $\Delta t = 5\times10^{-4}$.

(a) Z &K

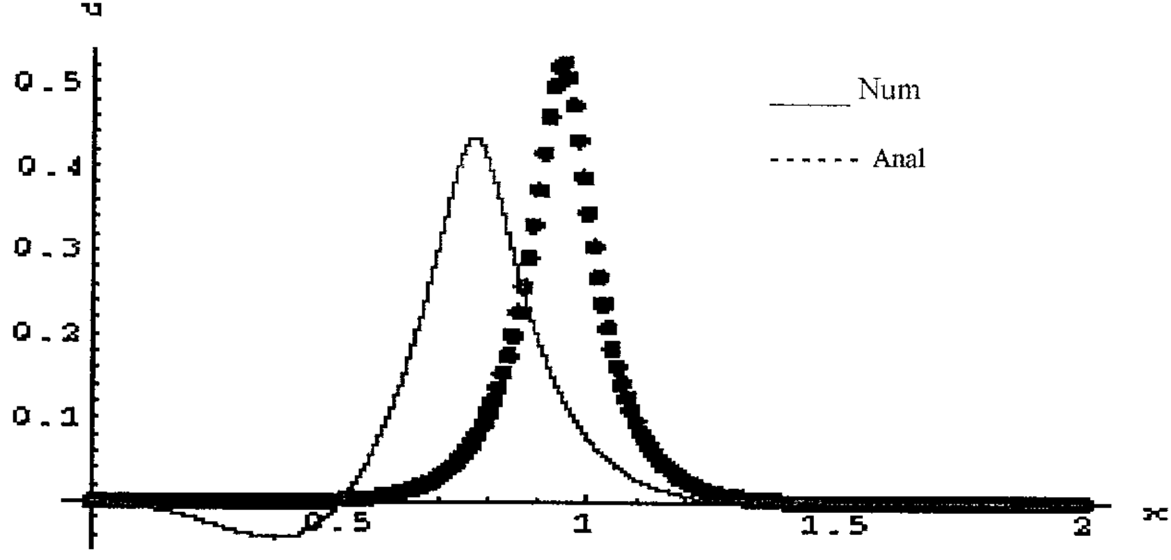
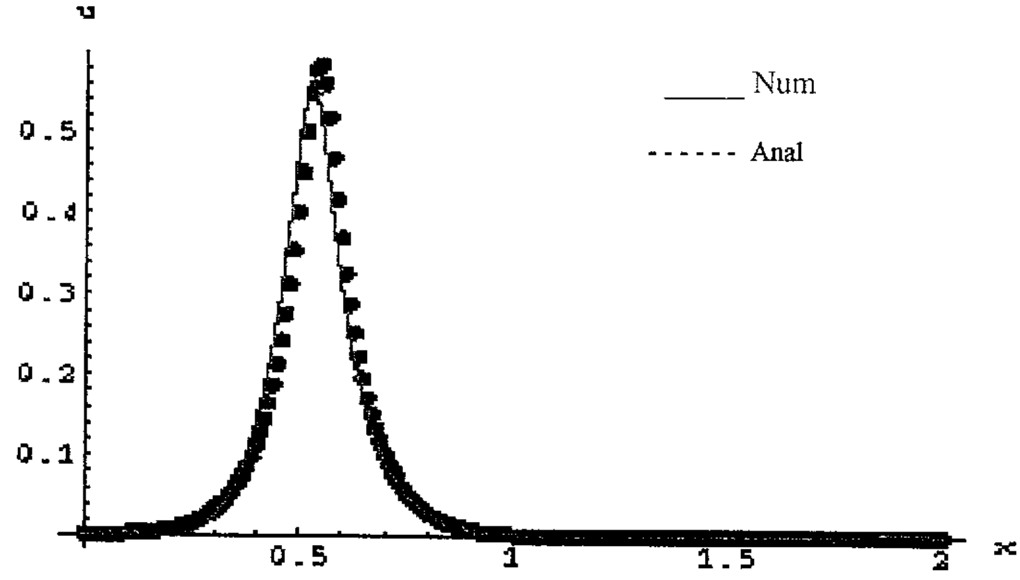


Fig (3.23) represents the analytical and numerical solutions for $p = 5$, and $t = 6$ when $v = 0.08$, $\Delta t = 5 \times 10^{-4}$.

(a) Z &K



(b) Hopscotch

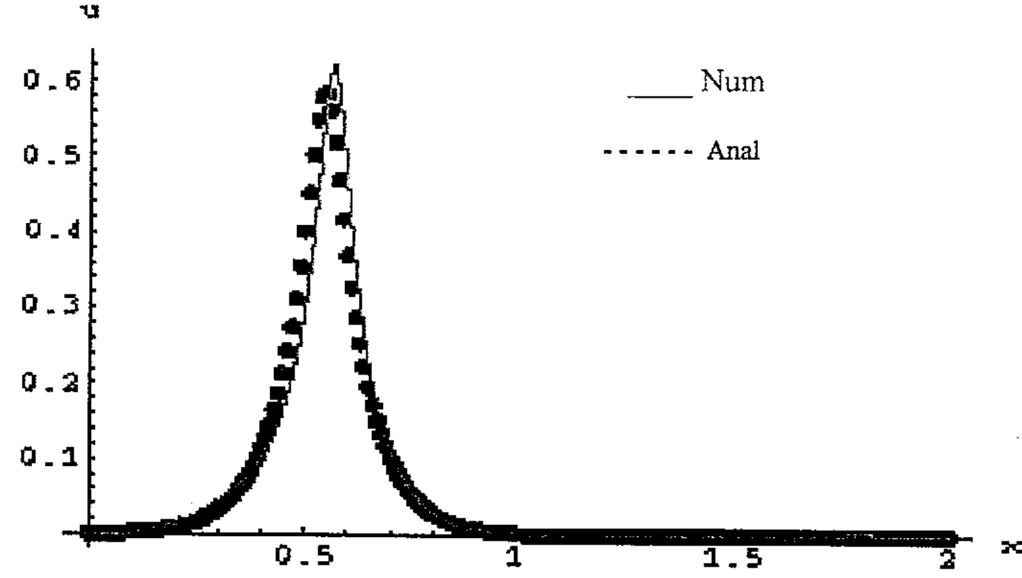


Fig (3.24) represents the analytical and numerical solutions for $p = 6$, and $t = 1$ when $v = 0.08$, $\Delta t = 5\times10^{-4}$.

(a) Z &K

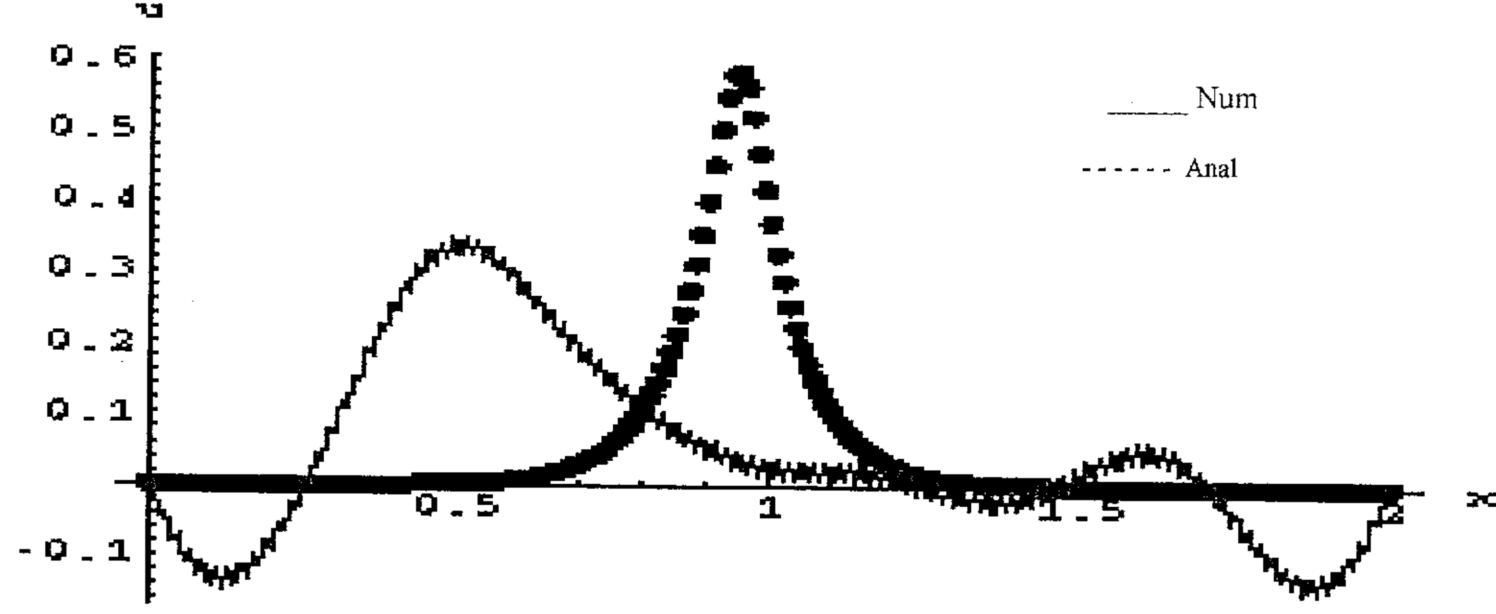


Fig (3.25) represens the analytical and numerical solutions $p = 6$, and $t = 6$ when $v = 0.08$, $\Delta t = 5\times10^{-4}$.

In the same direction, the stability condition for each method corresponding to p = 1,2 ,3 ,4 ,5, 6 at v = 0.04 are satisfied and the solitary waves which are results from the analytical and numerical solutions for each scheme are coincided as in table (7).

Table (7) Computing L_∞ beginning at T = 0.0 and ending at T = 6.0 with v = 0.04 and $\Delta t = 5\times10^{-4}$

Value of p	Mesh size		Time	L_∞^*	
	Δt^*	Δx^*		Z&K method	Hopscotch method
1 (Kdv) equation	5×10^{-4}	0.01	0.25	0.000206	0.000207
			0.50	0.000223	0.000235
			0.75	0.000206	0.000216
			1.0	0.000215	0.000193
			3.0	0.000225	0.000219
			6.0	0.000253	0.000202
2 (Mkdv) equation	5×10^{-4}	0.01	0.25	0.000735	0.000736
			0.50	0.000799	0.000839
			0.75	0.000724	0.000758
			1.0	0.000777	0.000696
			3.0	0.000776	0.000759
			6.0	0.000902	0.000717
3	5×10^{-4}	0.01	0.25	0.001115	0.001123
			0.50	0.001250	0.001303
			0.75	0.001103	0.001157
			1.0	0.001209	0.001112
			3.0	0.001221	0.001106
			6.0	0.002435	0.001925
4	5×10^{-4}	0.01	0.25	0.001368	0.001395
			0.50	0.001591	0.001655
			0.75	0.001379	0.001436
			1.0	0.001506	0.001439
			3.0	0.004128	0.003103
			6.0	0.009900	0.007441
5	5×10^{-4}	0.01	0.25	0.001589	0.001602
			0.50	0.002400	0.001941
			0.75	0.002932	0.002693
			1.0	0.003844	0.002730
			4.0	0.021823	0.020135
			6.0	0.043527	0.043205
6	5×10^{-4}	0.01	0.25	0.002708	0.002075
			0.50	0.004884	0.003634
			0.75	0.006824	0.005637
			1.0	0.009224	0.006886
			2.0	0.024180	0.021972
			4.0	0.086174	0.140915
			6.0	0.190666	Unstable

Also, at $v = 0.02$, $\Delta x = 0.01$ and $\Delta t = 5\times10^{-4}$, we are find that the two schemes (explicit and implicit) are satisfying the stability condition and the two solutions numerical and analytical are coincided for each method as indicated in table (8).

Table (8) Computing L_∞ beginning at T = 0.0 and ending at T = 6.0 with v = 0.02 and $\Delta t = 5\times10^{-4}$

Value of p	Mesh size		Time	L_∞^*	
	Δt^*	Δx^*		Z&K method	Hopscotch method
1 (Kdv) equation	5×10^{-4}	0.01	0.25	0.000222	0.000160
			0.50	0.000267	0.000187
			0.75	0.000283	0.000194
			1.0	0.000320	0.000204
			3.0	0.000457	0.000306
			6.0	0.000478	0.000318
2 (Mkdv) equation	5×10^{-4}	0.01	0.25	0.001115	0.000803
			0.50	0.001339	0.000940
			0.75	0.001420	0.000971
			1.0	0.001604	0.001020
			3.0	0.002288	0.001534
			6.0	0.002372	0.001598
3	5×10^{-4}	0.01	0.25	0.001906	0.001372
			0.50	0.002288	0.001609
			0.75	0.002424	0.001682
			1.0	0.002747	0.001740
			3.0	0.003945	0.002606
			6.0	0.004165	0.002784
4	5×10^{-4}	0.01	0.25	0.002494	0.001794
			0.50	0.002988	0.002102
			0.75	0.003162	0.002237
			1.0	0.003595	0.002259
			3.0	0.005205	0.003431
			6.0	0.005683	0.003778
5	5×10^{-4}	0.01	0.25	0.002935	0.002109
			0.50	0.003507	0.002461
			0.75	0.003704	0.002677
			1.0	0.004221	0.002636
			3.0	0.006022	0.004036
			6.0	0.006802	0.004650
6	5×10^{-4}	0.01	0.25	0.003283	0.002355
			0.50	0.003902	0.002726
			0.75	0.004140	0.003041
			1.0	0.004692	0.002956
			3.0	0.006521	0.004552
			6.0	0.013074	0.011888

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Appendix(A)

To obtain the truncation error of Zabusky & Kruskal scheme (3.3.1), the advection speed u, in the nonlinear term $\left(u^{p}\right)$ of equation (1.2.1) is assumed a locally constant as well as in the study of the stability analysis in supsection (3.3-2). Then the scheme (3.3.1) is rewrite in the form

$$u_n^{m+1}-u_n^{m-1}+r(p+1)(p+2)\bar{u}^{p}\left(u_{n+1}^{m}-u_{n-1}^{m}\right)+\frac{\gamma r}{h^2}\left[\left(u_{n+2}^{m}-u_{n-2}^{m}\right)-2\left(u_{n+1}^{m}-u_{n-1}^{m}\right)\right]=0 \quad (3.3.2)$$

Upon applying the Taylor expansion, the truncation error of the scheme (3.3.1) or (3.3.2) is illustrate as:

$$u_n^m+\tau u_t\Big|_n^m+\frac{\tau^2}{2!}u_{2t}\Big|_n^m+\frac{\tau^3}{3!}u_{3t}\Big|_n^m+\ldots-\left(u_n^m-\tau u_t\Big|_n^m+\frac{\tau^2}{2!}u_{2t}\Big|_n^m-\frac{\tau^3}{3!}u_{3t}\Big|_n^m+\ldots\right)$$

$$+2r(p+1)(p+2)\bar{u}^{p}\left(hu_x\Big|_n^m+\frac{h^3}{3!}u_{3x}\Big|_n^m+\frac{h^5}{5!}u_{5x}\Big|_n^m+\ldots\ldots\ldots\ldots\right)$$

$$+2r\frac{\gamma}{h^2}\left[2hu_x\Big|_n^m+\frac{(2h)^3}{3!}u_{3x}\Big|_n^m+\frac{(2h)^5}{5!}u_{5x}\Big|_n^m+\ldots\ldots\ldots\ldots-2(hu_x\Big|_n^m+\frac{h^3}{3!}u_{3x}\Big|_n^m\right.$$

$$\left.+\frac{h^5}{5!}u_{5x}\Big|_n^m+\ldots\ldots\ldots\ldots)\right]=0$$

$$2\tau\left[\left[u_t+\frac{\tau^2}{6}u_{3t}+\ldots\ldots\right]_n^m+(p+1)(p+2)\bar{u}^{p}\left[u_x+\frac{h^2}{6}u_{3x}+\ldots\ldots\right]_n^m\right.$$

$$\left.+\gamma\left[u_{3t}+\frac{30\ h^2}{5!}u_{5x}+\ldots\ldots\ldots\right]_n^m\right]=0$$

$$\left[u_t+(p+1)(p+2)\bar{u}^{p}u_x+\gamma u_{3x}\right]_n^m+\frac{\tau^2}{6}u_{3t}\Big|_n^m h^2\left[\frac{(p+1)(p+2)}{6}\bar{u}^{p}u_{3x}+\frac{30\gamma}{5!}u_{5x}\right]_n^m+\ldots\ldots=0$$

Then the truncation error of the scheme (3.3.1) is of order $\left[o\left(\tau^2\right)+o\left(h^2\right)\right]$.

To obtain the truncation error of the Hopscotch scheme (3.4.25), the a dvection speed u, in the nonlinear term (u^p) of the equation (1.2.1), is assumed a locally constant as well as in the study of the stability analysis in supsection (3.4-3). Upon using the Taylor series, hence, equation (3.4.25) rewrite as

$$[1+\lambda H_{2x}]u_n^{m+1} - [1-\lambda H_{2x}]u_n^{m-1} + [r(p+1)(p+2)\bar{u}^p - 4\lambda]H_x u_n^m = 0$$

where $\lambda = \dfrac{r\gamma}{2h^2}$

$$u_n^{m+1} - u_n^{m-1} + \lambda[u_{n+2}^{m+1} - u_{n-2}^{m+1}] + \lambda[u_{n+2}^{m-1} - u_{n-2}^{m-1}] + [r(p+1)(p+2)\bar{u}^p - 4\lambda][u_{n+1}^m - u_{n-1}^m] = 0$$

$$u\Big|_n^m + \tau u_t\Big|_n^m + \frac{\tau^2}{2!}u_{2t}\Big|_n^m + \frac{\tau^3}{3!}u_{3t}\Big|_n^m + \ldots - \left(u\Big|_n^m - \tau u_t\Big|_n^m + \frac{\tau^2}{2!}u_{2t}\Big|_n^m - \frac{\tau^3}{3!}u_{3t}\Big|_n^m + \ldots\right)$$

$$+\lambda\Bigg[u\Big|_n^m + \left(\tau u_t\Big|_n^m + 2hu_x\Big|_n^m\right) + \frac{1}{2!}\left(\tau\frac{\partial}{\partial t} + 2h\frac{\partial}{\partial x}\right)\left(\tau u_t\Big|_n^m + 2hu_x\Big|_n^m\right) +$$

$$\frac{1}{3!}\left(\tau\frac{\partial}{\partial t} + 2h\frac{\partial}{\partial x}\right)^2\left(\tau u_t\Big|_n^m + 2hu_x\Big|_n^m\right) + \ldots\ldots$$

$$-\left(u_n^m + \left(\tau u_t\Big|_n^m - 2hu_x\Big|_n^m\right) + \frac{1}{2!}\left(\tau\frac{\partial}{\partial t} - 2h\frac{\partial}{\partial x}\right)\left(\tau u_t\Big|_n^m - 2hu_x\Big|_n^m\right)\right.$$

$$\left.+\frac{1}{3!}\left(\tau\frac{\partial}{\partial t} - 2h\frac{\partial}{\partial x}\right)^2\left(\tau u_t\Big|_n^m - 2hu_x\Big|_n^m\right) + \ldots\right)\Bigg] +$$

$$+\lambda\Bigg[u_n^m + \left(-\tau u_t\Big|_n^m + 2hu_x\Big|_n^m\right) + \frac{1}{2!}\left(-\tau\frac{\partial}{\partial t} + 2h\frac{\partial}{\partial x}\right)\left(-\tau u_t\Big|_n^m + 2hu_x\Big|_n^m\right) +$$

$$\frac{1}{3!}\left(-\tau\frac{\partial}{\partial t} + 2h\frac{\partial}{\partial x}\right)^2\left(-\tau u_t\Big|_n^m + 2hu_x\Big|_n^m\right)\ldots\ldots$$

$$-\left(u_n^m + \left(-\tau u_t\Big|_n^m - 2hu_x\Big|_n^m\right) + \frac{1}{2!}\left(-\tau\frac{\partial}{\partial t} - 2h\frac{\partial}{\partial x}\right)\left(-\tau u_t\Big|_n^m - 2hu_x\Big|_n^m\right) +\right.$$

$$\left.\frac{1}{3!}\left(-\tau\frac{\partial}{\partial t} - 2h\frac{\partial}{\partial x}\right)^2\left(-\tau u_t\Big|_n^m - 2hu_x\Big|_n^m\right) + \ldots\right)\Bigg]$$

$$+\left[r(p+1)(p+2)\bar{u}^p-4\lambda\right]\left[u_n^m+hu_x\Big|_n^m+\frac{h^2}{2!}u_{2x}\Big|_n^m+\frac{h^3}{3!}u_{3x}\Big|_n^m+\ldots\ldots\right.$$

$$\left.-\left(u_n^m-hu_x\Big|_n^m+\frac{h^2}{2!}u_{2x}\Big|_n^m-\frac{h^3}{3!}u_{3x}\Big|_n^m+\ldots\right)\right]=0$$

$$2\tau\left\{u_t+\frac{\tau^2}{3!}u_{3t}+\ldots\right\}+\lambda\left[\left(2hu_x+\frac{1}{2!}2h\tau u_{xt}+\frac{1}{3!}(\tau^2\frac{\partial^2}{\partial t^2}+4h\tau\frac{\partial^2}{\partial t\partial x}+4h^2\frac{\partial^2}{\partial x^2})(\tau u_t+2hu_x)+\ldots\right)-\right.$$

$$\left.\left(-2hu_x-\frac{1}{2!}2h\tau u_{xt}+\frac{1}{3!}(\tau^2\frac{\partial^2}{\partial t^2}-4h\tau\frac{\partial^2}{\partial t\partial x}+4h^2\frac{\partial^2}{\partial x^2})(\tau u_t-2hu_x)+\ldots\ldots\right)\right]+$$

$$+\lambda\left[\left(2hu_x-\frac{1}{2!}2h\tau u_{xt}+\frac{1}{3!}(\tau^2\frac{\partial^2}{\partial t^2}-4h\tau\frac{\partial^2}{\partial t\partial x}+4h^2\frac{\partial^2}{\partial x^2})(-\tau u_t+2hu_x)+\ldots\ldots\right)-\right.$$

$$\left.\left(-2hu_x+\frac{1}{2!}2h\tau u_{xt}+\frac{1}{3!}(\tau^2\frac{\partial^2}{\partial t^2}+4h\tau\frac{\partial^2}{\partial t\partial x}+4h^2\frac{\partial^2}{\partial x^2})(-\tau u_t-2hu_x)+\ldots\ldots\right)\right]+$$

$$+2\left[r(p+1)(p+2)\bar{u}^p-4\lambda\right]\left[hu_x+\frac{h^3}{3!}u_{3x}+\ldots\ldots\ldots\right]=0$$

$$2\tau\left\{u_t+\frac{\tau^2}{3!}u_{3t}+\ldots\ldots\right\}+\frac{\tau\gamma}{2h^3}\left[8hu_x+\frac{32}{3!}h^3u_{3x}+\frac{24}{3!}\tau^2hu_{xtt}\right]+$$

$$\frac{2\tau}{h}(p+1)(p+2)\bar{u}^p\left[hu_x+\frac{h^3}{3!}u_{3x}+\ldots\ldots\ldots\right]-4\frac{\tau\gamma}{h^3}\left[hu_x+\frac{h^3}{3!}u_{3x}+\ldots\ldots\right]=0$$

$$2\tau\left\{u+(p+1)(p+2)u^pu_x+\gamma u_{3x}\right\}+2\tau\left\{\frac{\tau^2}{3!}u_{3t}+\frac{\tau^2}{h^2}u_{ttx}+\frac{h^2}{3}(p+1)(p+2)\bar{u}^pu_{3x}+\ldots\ldots\ldots\right\}=0$$

Then the truncation error is order $\tau o(\tau^2+\frac{\tau^2}{h^2}+h^2)$.

Appendix (B)

THE FIRST FEW COEFFICIENTS OF (RR) FOR P = 1 (a_1 is arbitrary)
WHEN $C_1 = 0$, h = 0

$$(n \hat{} 2 - 1) * a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m \qquad \text{for } n \geq 2$$

=== === === === === === === === === === === === === ==

Do[Print[a[i] = a_i], {i, 1, 10}]

a_1

a_2

a_3

a_4

a_5

a_6

a_7

a_8

a_9

a_{10}

$\text{Table}\left[-\sum_{m=1}^{n-1} a[n-m] * a[m] / (n \hat{} 2 - 1), \{n, 2, 10\}\right];$

n = 2

2

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \hat{} 2 - 1)\right]$

$-\frac{a_1^2}{3}$

n = 3

3

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \hat{} 2 - 1)\right]$

$\frac{a_1^3}{12}$

n = 4

4

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \wedge 2 - 1)\right]$

$-\frac{a_1^4}{54}$

n = 5

5

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \wedge 2 - 1)\right]$

$\frac{5 a_1^5}{1296}$

n = 6

6

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \wedge 2 - 1)\right]$

$-\frac{a_1^6}{1296}$

n = 7

7

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \wedge 2 - 1)\right]$

$\frac{7 a_1^7}{46656}$

n = 8

8

$\text{Print}\left[a_n = -\sum_{m=1}^{n-1} a_{n-m} * a_m / (n \wedge 2 - 1)\right]$

$-\frac{a_1^8}{34992}$

THE FIRST FEW COEFFCIENTS OF (RR) FOR P = 2 (a_1 is arbitrary)
WHEN $C_1 = 0$, $h = 0$

$$(n\hat{}2-1)*a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m}*a_{m-l}*a_l$$

$$\text{Table}\left[-\sum_{m=1}^{n-1} a[n-m]*a[m]/(n\hat{}2-1),\ \{n, 2, 10\}\right];$$

n = 2

2

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m}*a_{m-l}*a_l/(n\hat{}2-1)\right]$$

0

n = 3

3

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m}*a_{m-l}*a_l/(n\hat{}2-1)\right]$$

$$-\frac{a_1^3}{8}$$

n = 4

4

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m}*a_{m-l}*a_l/(n\hat{}2-1)\right]$$

0

n = 5

5

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m}*a_{m-l}*a_l/(n\hat{}2-1)\right]$$

$$\frac{a_1^5}{64}$$

n = 6

6

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m} * a_{m-l} * a_l / (n^2 - 1)\right]$$

0

n = 7

7

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m} * a_{m-l} * a_l / (n^2 - 1)\right]$$

$$-\frac{a_1^7}{512}$$

n = 8

8

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m} * a_{m-l} * a_l / (n^2 - 1)\right]$$

0

n = 9

9

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m} * a_{m-l} * a_l / (n^2 - 1)\right]$$

$$\frac{a_1^9}{4096}$$

n = 10

10

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m} * a_{m-l} * a_l / (n^2 - 1)\right]$$

0

n = 11

11

$$\text{Print}\left[a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m} * a_{m-l} * a_l / (n^2 - 1)\right]$$

$$-\frac{a_1^{11}}{32768}$$

THE FIRST FEW COEFFICIENTS OF (RR) FOR P = 3 (a_1 is arbitrary)
WHEN $C_1 = 0$, $h = 0$

$$(n\wedge 2-1)\, a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r$$

for $n \geq 4$

=== === === === === === === === === === === === === === === === === ===

$$\text{Table}\left[-\sum_{m=1}^{n-1} a[n-m] * a[m] / (n\wedge 2-1),\ \{n, 2, 20\}\right];$$

$a_2 = a_3 = 0$

0

n = 4

4

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2-1)\right]$$

$$-\frac{a_1^4}{15}$$

n = 5

5

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2-1)\right]$$

0

n = 6

6

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2-1)\right]$$

0

n = 7

7

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2 - 1)\right]$$

$$\frac{a_1^7}{180}$$

n = 8

8

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2 - 1)\right]$$

0

n = 9

9

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2 - 1)\right]$$

0

n = 10

10

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n\wedge 2 - 1)\right]$$

$$-\frac{a_1^{10}}{2025}$$

n = 11

11

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n \wedge 2 - 1)\right]$$

0

n = 12

12

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n \wedge 2 - 1)\right]$$

0

n = 13

13

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n \wedge 2 - 1)\right]$$

$$\frac{11\, a_1^{13}}{243000}$$

$a_{14} = a_{15} = 0$

0

n = 16

16

$$\text{Print}\left[a_n = -\sum_{m=3}^{n-1}\sum_{l=2}^{m-1}\sum_{r=1}^{l-1} a_{n-m} * a_{m-l} * a_{l-r} * a_r / (n \wedge 2 - 1)\right]$$

$$-\frac{77\, a_1^{16}}{18225000}$$

THE FIRST FEW COEFFICIENT OF (RR) at $p = 4$ (a_1 arbitrary)
WHEN $C_1 = 0$, $h = 0$

$$(n\wedge 2-1)*a_n+\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1}a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_k=0 \qquad \text{for } n\geq 5$$

$a_2 = a_3 = a_4 = 0$

0

$n = 5$

5

$$\text{Print}\left[a_n=-\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1}a_{n-m}\,a_{m-l}*a_{l-r}*a_{r-k}*a_k/(n\wedge 2-1)\right]$$

$$-\frac{a_1^5}{24}$$

$n = 6$

6

$$\text{Print}\left[a_n=-\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1}a_{n-m}\,a_{m-l}*a_{l-r}*a_{r-k}*a_k/(n\wedge 2-1)\right]$$

0

$n = 7$

7

$$\text{Print}\left[a_n=-\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1}a_{n-m}\,a_{m-l}*a_{l-r}*a_{r-k}*a_k/(n\wedge 2-1)\right]$$

0

$n = 8$

8

$$\text{Print}\left[a_n=-\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1}a_{n-m}\,a_{m-l}*a_{l-r}*a_{r-k}*a_k/(n\wedge 2-1)\right]$$

0

$n = 9$

9

$$-\text{Print}\left[a_n=-\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1}a_{n-m}\,a_{m-l}*a_{l-r}*a_{r-k}*a_k/(n\wedge 2-1)\right]$$

$$\frac{a_1^9}{384}$$

n = 10

10

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

0

n = 11

11

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

0

n = 12

12

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

0

n = 13

13

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

$$-\frac{5 a_1^{13}}{27648}$$

n = 14

14

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

0

n = 15

15

$$\text{Print}\left[a_n = \sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

0

$n = 16$

16

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

0

$n = 17$

17

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

$$\frac{35\, a_1^{17}}{2654208}$$

$a_{18} = a_{19} = a_{20} = 0$

0

$n = 21$

21

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

$$-\frac{7\, a_1^{21}}{7077888}$$

$a_{22} = a_{23} = a_{24} = 0$

0

$n = 25$

25

$$\text{Print}\left[a_n = -\sum_{m=4}^{n-1}\sum_{l=3}^{m-1}\sum_{r=2}^{l-1}\sum_{k=1}^{r-1} a_{n-m}\, a_{m-l} * a_{l-r} * a_{r-k} * a_k / (n\,{}^\wedge 2 - 1)\right]$$

$$\frac{77\, a_1^{25}}{1019215872}$$

THE FIRST FEW COEFFICIENTS OF (RR) at $p = 5$ (a_1 arbitrary)
WHEN $C_1 = 0$, $h = 0$

$$(n^2-1)*a_n + \sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_s = 0$$

for $n \geq 6$

=== === === === === === === === === === === === === === === === ==

$a_2 = a_3 = a_4 = a_5 = 0$

0

$n = 6$

6

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_s/(n^2-1)\right]$$

$-\frac{a_1^6}{35}$

$n = 7$

7

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_s/(n^2-1)\right]$$

0

$n = 8$

8

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_s/(n^2-1)\right]$$

0

$n = 9$

9

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_s/(n^2-1)\right]$$

0

n = 10

10

Print$\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$

0

n = 11

11

Print$\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$

$\frac{a_1^{11}}{700}$

n = 12

12

Print$\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$

0

n = 13

13

Print$\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$

0

n = 14

14

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$$

0

n = 15

15

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$$

0

n = 16

16

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$$

$$-\frac{a_1^{16}}{12250}$$

n = 17

17

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$$

0

n = 18

18

$$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$$

0

$a_{19} = a_{20} = 0$

0

$n = 21$

21

$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$

$\frac{17\, a_1^{21}}{3430000}$

$a_{22} = a_{23} = a_{24} = a_{25} = 0$

0

$n = 26$

26

$\text{Print}\left[a_n = -\sum_{m=5}^{n-1}\sum_{l=4}^{m-1}\sum_{r=3}^{l-1}\sum_{k=2}^{r-1}\sum_{s=1}^{k-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_s / (n\text{^}2 - 1)\right]$

$-\frac{187\, a_1^{26}}{600250000}$

THE FIRST FEW COEFFICIENTS OF (RR) AT $p = 6$ (a_1 arbitrary)
WHEN $C_1 = 0$, $h = 0$

$$(n^2-1)*a_n + \sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_{s-h}*a_h = 0$$

for $n \geq 7$

= = = = = = = = = = = = = = = = === === === === ===

$a_2 = a_3 = a_4 = a_5 = a_6 = 0$

0

n = 7

7

$$\text{Print}\Big[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_{s-h}*a_h/(n^2-1)\Big]$$

$-\frac{a_1^7}{48}$

n = 8

8

$$\text{Print}\Big[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_{s-h}*a_h/(n^2-1)\Big]$$

0

n = 9

9

$$\text{Print}\Big[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m}*a_{m-l}*a_{l-r}*a_{r-k}*a_{k-s}*a_{s-h}*a_h/(n^2-1)\Big]$$

0

n = 10

10

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\wedge 2 - 1)\right]$$

0

n = 11

11

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\wedge 2 - 1)\right]$$

0

n = 12

12

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\wedge 2 - 1)\right]$$

0

n = 13

13

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\wedge 2 - 1)\right]$$

$$\frac{a_1^{13}}{1152}$$

n = 14

14

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\wedge 2 - 1)\right]$$

0

n = 15

15

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n^2 - 1)$]

0

n = 16

16

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n^2 - 1)$]

0

n = 17

17

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n^2 - 1)$]

0

n = 18

18

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n^2 - 1)$]

0

n = 19

19

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\text{^}2 - 1)\right]$$

$$-\frac{7 a_1^{19}}{165888}$$

n = 20

20

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\text{^}2 - 1)\right]$$

0

n = 21

21

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\text{^}2 - 1)\right]$$

0

n = 22

22

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\text{^}2 - 1)\right]$$

0

n = 23

23

$$\text{Print}\left[a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\text{^}2 - 1)\right]$$

0

n = 24

24

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\string^2 - 1)$]

0

n = 25

25

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\string^2 - 1)$]

$$\frac{35 a_1^{25}}{15925248}$$

$a_{26} = a_{27} = a_{28} = a_{29} = a_{30} = 0$

0

n = 31

31

Print[$a_n = -\sum_{m=6}^{n-1}\sum_{l=5}^{m-1}\sum_{r=4}^{l-1}\sum_{k=3}^{r-1}\sum_{s=2}^{k-1}\sum_{h=1}^{s-1} a_{n-m} * a_{m-l} * a_{l-r} * a_{r-k} * a_{k-s} * a_{s-h} * a_h / (n\string^2 - 1)$]

$$-\frac{91 a_1^{31}}{764411904}$$

Appendix(C)

In this appendix we have written two fortran programs, the first program shows how the Generalized Kdv equation (1.2.1) is solved numericaly by using Zabusky & Kruskal method and the second one shows how this equation is solved numericaly by using the Hopscotch method where the stability condition and L_∞error are calculated for each method for p =1,2,3,4,5,6 at finite time, where the analytical solution is

$$u(x,t)=\left(\frac{v}{2}\right)^{\frac{1}{p}}\operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}\left(\sqrt{\frac{v}{\gamma}}(x-vt)+\Delta\right)\right),$$

and the numerical solution is

$$u_n^{m+1}=u_n^{m-1}-\frac{(p+1)(p+2)r}{3^p}\left(u_{n+1}^m+u_n^m+u_{n-1}^m\right)^p\left(u_{n+1}^m-u_{n-1}^m\right)-\frac{\gamma r}{h^2}\left(u_{n+2}^m-2u_{n+1}^m+2u_{n-1}^m-u_{n-2}^m\right), \text{ (Z\&K method)}$$

while the numerical solution by Hopscotch method is

$$u_n^{m+1}=u_n^m-\frac{r}{2}(p+1)(p+2)\left[F_{n+1}^m-F_{n-1}^m\right]-\frac{r\gamma}{2h^2}\left[u_{n+1}^m-2u_{n+2}^m+2u_{n-1}^m-u_{n-2}^m\right]. \text{ (explicit)}$$

$$u_n^{m+1}+\frac{r\gamma}{2h^2}\left(u_{n+2}^{m+1}-u_{n-2}^{m+1}\right)=u_n^m-\frac{r}{2}(p+1)(p+2)\left(F_{n+1}^{m+1}-F_{n-1}^{m+1}\right)+\frac{r\gamma}{h^2}\left(u_{n+1}^{m+1}-u_{n-1}^{m+1}\right) \text{ (implicit)}$$

Now we are difine some of the variables which used in this programs as the following

The variable	Definition
M	Represent the parameter p
C	represent the velocity v
DX	the increment in space (h)
DT	the increment in TIME (τ)
DTDDX	$\frac{\tau}{h}=r$
RJDTH2	$\frac{r\gamma}{2h^2}$
SB	The wave number $K=\sqrt{\frac{v}{\gamma}}$
SC	Stability condition
BETA	The factor of nonlinear term (p+1)(p+2)
HRB	$\frac{r.beta}{2}$
D	$D=p.\Delta/2$, Δ=-6 (The phase shift)
UN	Initial condition
ZUC	u_n^m
ZUE	u_{n+1}^m
ZUW	u_{n-1}^m
ZUEE	u_{n+2}^m
ZUWW	u_{n-2}^m
ZUNP1	u_n^{m+1}
UNM1	u_n^{m-1}
ZUNUM	the numerical solution
ZUANAL	the analytical solution

The first program

```
      THIS PROGRAM IS MAKE TO SOLVE THE GENERALIZED KDV EQUATION
      NUMRRICALLY BY USING Z&K METHOD WITH USING TAHA STABILITY
      FOR P=1,2,3,4,5,6,AND CALCULATION L(INFINITY) AT FINITE TIME FOR
      EACH VALUE OF P
      THE GENERAL FORM OF THE GENERALIZED KDV EQUATION IS
```

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{3x} = 0$$

```
      ***************************************************************
      INTEGER I,IS1,IF1,K,I1,I11,I2,I3,ISM2,IFP2,NSTEP,IC,IE,IW,IEE,K1,
     *  IWW,N,IF2,IS2,NSTEPR,NSTEPF,NSTPD2,NSTEPD,M,DM,NSTEPS

      real UN,UNP1,A,G,S,R,ZUANA,IIS,H1,IIF,C,D,JAMA,BETA,C1,DX,DT,DXS,
     *  T,SB,PROD,RATIO,A1,G1,B1,X,ARG,DTDDX,HRB,RJDTH2,UNM1,H,Y,UNP,AD,
     *  ZUC,ZUE,ZUW,ZUS,ZUM,ZUEE,C3,ZUWW,ZFLUX1,SECHS,A5,A11,A2,A3,AZ,B,
     *  ZFLUX2,ZUNP1,DMBT,ZBT,ZB,ZUMAX,ZUANAL,H3,D3,D4,ZUNUM,SC,V,APROD,
     *  MAX,DD,SSE,DXI,ZSS
      ----------------------------------------------------------------
      DIMENSION UN(210),UNP1(210),UNM1(210),ZUANA(210),UNP(210)
      OPEN(6,FILE='KZN.RES')
      OPEN(5,FILE='H3.RES')
      ----------------------------------------------------------------
      WRITE (5,1)
 1    FORMAT(3X,'COMPUTING LINFINITY BEGINNING  T=0 & ENDING AT T=4')
      WRITE(5,2)
 2    FORMAT('V=0.3 OF ZABUSKY&KRUSKAL METHOD TO SOLVE GKDV EQUATION')
      WRITE(5,3)
 3    FORMAT('-----------------------------------------------------')
      WRITE(5,4)
 4    FORMAT(16X,'**MESHSIZE**')
      WRITE(5,5)
 5    FORMAT(3x,'VALUE P',7X,'DX',6X,'DT',10X,'TIME',7X,'L(INFINITY)')
      ----------------------------------------------------------------
      M=1
 6    WRITE(5,3)
      NSTEPF=8000
      NSTEPS=500
      WRITE(6,7)M
 7    FORMAT(20X,'M=',I5)
      IIS=3
      IIF=199
      PRINCIPLE CONSTANT
      -----------------
      C=0.3
      JAMA=4.84E-04
      DX=0.01
      DXS=DX*DX
      NSTEP=1
      NSTEPR=2000
      DT=0.0005
      T=0.0
      SB=SQRT(C/JAMA)
      A1=M*SB/2.0
      B1=M*C*SB/2.0
      BETA=(M+1.0)*(M+2.0)
      A5=C/2.0
      H=1.0
      Y=H/M
      D=-6.0*M/2.0
      WRITE(5,8)M,DX,DT
 8    FORMAT(5X,I3,5X,F7.5,3X,F7.5)

      UNDER INITIAL CONDITION
      =======================
      DO 9,I=1,201
       UNM1(I)=0.0
 9    CONTINUE
      DXI=-DX
      X=DXI
```

```
      INITIAL CONDITION
      ================
      WRITE(6,10)
 10   FORMAT(20X,'POINT NUMBER',10X,'VALUE OF U')
      DO 100,I=1,201
      X=X+DX
       IF(M.EQ.4.OR.M.EQ.5.OR.M.EQ.6)THEN
        IF(I.GT.2.AND.I.LE.143)THEN
        ARG=A1*X+D
        SSE=SECHS(ARG)
        UN(I)=(A5*SSE)**(1.0/M)
        ELSE
        UN(I)=0.0
        ENDIF
       ELSE
        IF(I.EQ.1.OR.I.EQ.2.OR.I.EQ.201.OR.I.EQ.200)THEN
        UN(I)=0.0
        ELSE
        ARG=A1*X+D
        SSE=SECHS(ARG)
        UN(I)=(A5*SSE)**Y
        END IF
       ENDIF
      UNP1(I)=0.0
      WRITE(6,11)I,UN(I)
 11   FORMAT(20X,I5,17X,F10.5)
      PRINT*,I,UN(I)
 100  CONTINUE
      OUT PUT OF INITIAL CONDITION FOR ONE SINGAL SOLITON
      --------------------------------------------------
      DM=1
 200  T=T+DT
      DTDDX=DT/DX
      HRB=DTDDX*BETA
      RJDTH2=DTDDX*JAMA/DXS

      EXPLICIT CALCULATION OF KRUSKAL AND ZABUSKY SCHEME
      --------------------------------------------------
      WRITE(*,*)'EXPLICIT CACULATION OF Z&K SCHEME'
       DO 300,I=IIS,IIF
       IC=I
        IE=IC+1
        IEE=IE+1
        IW=IC-1
        IWW=IW-1
        ZUC=UN(IC)
        ZUE=UN(IE)
        ZUW=UN(IW)
         IF(NSTEP.EQ.1.0) THEN
         ZUM=UN(I)
         ELSE
         ZUM=UNM1(IC)
         END IF
        ZUS=(ZUE-ZUW)
        ZUEE=UN(IEE)
        ZUWW=UN(IWW)
        ZSS=(ZUE+ZUW+ZUC)/3.0
        ZFLUX1=HRB*ZSS**M
        ZFLUX2=RJDTH2*(ZUEE-2.0*ZUE+2.0*ZUW-ZUWW)
        ZUNP1=ZUM-(ZFLUX1*ZUS)-ZFLUX2
        UNP1(I)=ZUNP1
 300   CONTINUE

       DO 12,I=IIS,IIF
        UNM1(I)=UN(I)
         UN(I)=UNP1(I)
 12    CONTINUE
      CALCULATE THE TIME STEP
      ---------------------
      ZUMAX=0
       DO 13 I=IIS,IIF
```

```
        ZUNP1=UNP1(I)
        AZ=ABS(ZUNP1)
        IF(AZ.GT.ZUMAX) ZUMAX=ZUNP1
 13    CONTINUE
      PROD=-(BETA/3.0)*(ZUMAX**M)+JAMA/DXS
      APROD=ABS(PROD)
      IF(APROD.GT.0.0001) GO TO 15
      WRITE(6,14)APROD,ZUMAX
 14   FORMAT(10X,F12.5,5X,F10.5)
      STOP
 15    DO 16,I=IIS,IIF
        UN(I)=UNP1(I)
        UNP1(I)=0.0
 16   CONTINUE

      OUT PUT AT REGIRED TIME STEP
      ----------------------------
      IF(NSTEP.GT.NSTEPR)GOTO 17
       IF(((NSTEP/NSTEPS)*NSTEPR).EQ.(NSTEPR/NSTEPS)*NSTEP)GOTO 20
 17      IF(((NSTEP/NSTEPR)*NSTEPF).NE.(NSTEPF/NSTEPR)*NSTEP) GO TO 25

      WRITE(6,18)NSTEP,DT,T
 18   FORMAT(10X,'NSTEP=',I4,5X,'DT=',F8.5,5X,'T=',F8.5,10X)

      WRITE(6,19)
 19   FORMAT(10X,'POSITION',6X,'NUM SOL',10X,'ANAL ASOL')

      CACULATION THE ANALYTICAL AND NUMERICAL RESULTS
      -----------------------------------------------
 20   X=DXI
      ISM2=IIS-2
      IFP2=IIF+2
      DMBT=D-B1*T
       DO 22 ,I=ISM2,IFP2
        X=X+DX
        IF(M.EQ.4.OR.M.EQ.5.OR.M.EQ.6) THEN
          IF(I.GE.2.AND.I.LE.143)THEN
          ARG=A1*X+DMBT
          SSE=SECHS(ARG)
           ZUANA(I)=(A5*SSE)**Y
          ZUANAL=ZUANA(I)
          ELSE
          ZUANAL=0.0
          ENDIF
        ELSE
        ARG=A1*X*DMBT
        SSE=SECHS(ARG)
        ZUANA(I)=(A5*SSE)**(1.0/M)
        ZUANAL=ZUANA(I)
        END IF
        ZUNUM=UN(I)
        PRINT*,I,ZUNUM,ZUANAL
        WRITE(6,21)X,ZUNUM,ZUANAL
 21     FORMAT(10X,F8.5,5X,F12.5,10X,F10.5)
 22    CONTINUE

      CALCULATE THE VALUE OF L(INFINITY) AT EACH TIME
      -----------------------------------------------
      MAX=0.0
       DO 23,I=1,201
        DD=UN(I)-ZUANA(I)
        AD=ABS(DD)
        IF(AD.GT.MAX) MAX=AD
 23    CONTINUE
      WRITE(5,24)T,MAX
 24   FORMAT(38X,F9.6,6X,F9.6)
 25   SC=DTDDX*APROD
      NSTEP=NSTEP+1
     IF(NSTEP.GT.NSTEPF) GOTO 27
     G=3*(3**0.5)
     RATIO=2/G
```

```
      IF(SC.LE.RATIO) GOTO 200
      WRITE(6,26)SC
 26   FORMAT(10X,'SC=',F10.5)
 27   M=M+DM
        IF(M.GT.6) GOTO 28
        GOTO 6
 28     STOP
 29   CLOSE(6)
      CLOSE(5)
      END
      ------------------------------------------------------------------
      SUPROTINE TO CALCULATE SECH^2
      -----------------------------
      FUNCTION SECHS(X)
      REAL A11,A2,A3,A4
      A3=COSH(X)
      A4=1/A3
      SECHS=A4*A4
      RETURN
      END
      ================================================================
```

The second program

```
      THIS PROGRAM IS MAKE TO SOLVE THE GENERALIZED KDV EQUATION
      NUMERICALY BY USING THE HOPSCOTCH METHOD FOR P=1,2,3,4,5,6,AND
      CALCULATE L(INFINITY) AT FINITE TIME
      THE GENERAL FORM OF THE GENERALIZED KDV EQUATION IS
```

$$u_t + (p+1)(p+2)u^p u_x + \gamma u_{3x} = 0$$

```
      *****************************************************************
      INTEGER I,IS1,IF1,K,I1,I11,I2,I3,ISM2,IFP2,NSTEP,IC,IE,IW,IEE,
     * K1,IWW,N,IF2,IS2,NSTEPS,NSTEPF,G,NSTEPR,NSTPD2,NSTEPD,M,DM
      ------------------------------------------------------------
      REAL A1,B1,UN,UNP1,A,B,S,R,ZUANA,IIS,IIF,C,D,JAMA,DX,DXS,BETA,DT,
     * T,SB,G1,D1,D2,D3,C1,D5,X,ARG,DTDDX,HRB,RJDTH2,A5,ZUC,ZUE,ZUW,A4,
     * ZUES,ZUWS,d4,ZUEE,ZUWW,ZFLUX1,SECHS,A11,A2,A3,ZFLUX2,ZUNP1,DMBT,
     * ZBT,ZB,ZUMAX,ZUANAL,ZUNUM,SC,AZ,PROD,APROD,MAX,DD,AD,SSE,DXI,
      --------------------------------------------------------------
      DIMENSION UN(310),UNP1(310),A(310),B(310),S(310),R(310),ZUANA(310)
      OPEN(5,FILE='S4.RES')
      OPEN(6,FILE='sc.RES')
      ..............................................................
      WRITE(5,1)
 1    FORMAT(9X,'COMPUTING L INFINITY BEGINNING AT T=0 &ENDING AT T=4')
      WRITE(5,2)
 2    FORMAT(10X,'V=0.08 FOR HOPSCOTCH METHOD TO SOLVE G-KDV EQUATION')
      WRITE(5,3)
 3    FORMAT(3X,'--------------------------------------------------')
      WRITE(5,4)
 4    FORMAT(16X,'**MESH SIZE**')
      WRITE(5,5)
 5    FORMAT(3X,'VALUE OF P',4X,'DX',6X,'DT',8X,'TIME',7X,'L-INFINITY')

      M=1
 6    WRITE(*,*)M
      NSTEPF=8000
      WRITE(5,7)M
 7     FORMAT(10X,'M=',I5)

      IIS=3
      IIF=199

      PRINCIPLE CONSTANT
      ==================
      JAMA=4.84E-04
      THE FORMULA OF THE PHASE SHIFT
      D=-6.0*M/2.0
      ==========================================================
      DX=0.01
      DXI=-DX
```

```
      c=0.08
      DXS=DX*DX
      NSTEP=1
      NSTEPS=500
      NSTEPR=2000
      DT=0.0005
      T=0.0
      SB=SQRT(C/JAMA)
      A1=M*SB/2.0
      B1=M*C*SB/2.0
      BETA=(M+1.0)*(M+2.0)
      A5=C/2.0

      WRITE(5,8)M,DX,DT
 8    FORMAT(5X,I3,7X,F7.5,4X,F7.5)

      INITIAL CONDITION
      =================
      X=DXI
      WRITE(5,9)
 9    FORMAT(20X,'POINT NUMBER',10X,'VALUE OF U')
       DO 11,I=IIS-2,IIF+2
        X=X+DX
        IF(M.EQ.4.OR.M.EQ.5.OR.M.EQ.6)THEN
         IF(I.GT.2.AND.I.LE.143)THEN
         ARG=A1*X+D
         SSE=SECHS(ARG)
         UN(I)=(A5*SSE)**(1.0/M)
         ELSE
         UN(I)=0.0
         ENDIF
        ELSE
         IF(I.EQ.(IIS-2).OR.I.EQ.(IIS-1).OR.I.EQ.(IIF+2).OR.I.EQ.
     *   (IIF+1))THEN
         UN(I)=0.0
         ELSE
         ARG=A1*X+D
         SSE=SECHS(ARG)
         UN(I)=(A5*SSE)**(1.0/M)
         ENDIF
        ENDIF
       UNP1(I)=0.0
       WRITE(10,*)X,UN(I)
 10    FORMAT(20X,I5,16X,F14.6)
 11   CONTINUE


      OUT PUT OF INITIAL CNDITION FOR ONE SINGAL SOLITON

      DM=1
 12   T=T+DT
      IF(NSTEP.GT.NSTEPF) GOTO 30
      DTDDX=DT/DX
      HRB=0.5*DTDDX*BETA
      RJDTH2=DTDDX*JAMA/(2.0*DXS)
      NSTPD2=NSTEP/2.0
      NSTEPD=2*NSTPD2
      IS1=IIS+NSTEP-NSTEPD
      IF1=IIF-NSTEP+NSTEPD
      IS2=IIS+(NSTEP+1)-(((NSTEP+1)/2)*2)
      IF2=IIF-(NSTEP+1)+(((NSTEP+1)/2)*2)
      G=M+1

      EXPLICIT CALCULATION OF HOPSCOTCH SCHEME WHEN(I+M) IS EVEN

      WRITE(*,*)'EXPLICIT CACULATION WHEN I+M IS EVEN'
       DO 13,I=IS1,IF1,2
        IC=I
        IE=IC+1
        IEE=IE+1
        IW=IC-1
```

```
      IWW=IW-1
      ZUC=UN(IC)
      ZUE=UN(IE)
      ZUW=UN(IW)
      ZUES=(ZUE**G)/G
      ZUWS=(ZUW**G)/G
      ZUEE=UN(IEE)
      ZUWW=UN(IWW)
      ZFLUX1=(ZUES-ZUWS)*HRB
      ZFLUX2=RJDTH2*(ZUEE-2.0*ZUE+2.0*ZUW-ZUWW)
      ZUNP1=ZUC-ZFLUX1-ZFLUX2
      UNP1(I)=ZUNP1
 13   CONTINUE
     IMPLICIT CALCULATION OF HOPSCOTCH METHOD WHEN (I+M) IS ODD
     WRITE(*,*)'IMPLICIT CALCULATION WHEN I+M IS ODD'
     WRITE(*,*)'NSTEP=',NSTEP,'M=',M
      DO 14,I=IS2,IF2,2
       IC=I
       IE=IC+1
       IW=IC-1
       ZUC=UN(IC)
       ZUE=UNP1(IE)
       ZUW=UNP1(IW)
       ZUES=(ZUE**G)/G
       ZUWS=(ZUW**G)/G
       ZFLUX1=HRB*(ZUES-ZUWS)
       ZFLUX2=2.0*RJDTH2*(ZUE-ZUW)
       ZBT=ZUC-ZFLUX1+ZFLUX2
       B(I)=ZBT
 14   CONTINUE

     THE ARRAY B IS THE RHS IN AU=B RE-ARRANG THE ELEMENTS INSIDE B
     ----------------------------------------------------------------
     K=(IF2-IS2)/2+1
     IF((K/2)*2.EQ.K) K1=2
     IF((K/2)*2.NE.K) K1=1
      DO 15,I1=1,K
       A(I1)=-RJDTH2
       R(I1)=1.0
       S(I1)=RJDTH2
       I11=2*I1+K1
       B(I1)=B(I11)
 15   CONTINUE
     N=K

     CALLING SUBROUTINE TO FIND VALUE OF ELEMENTS IN TRIDIOGNAL MATRIX

     CALL TRID(A,R,S,B,N)

     RE-ARRANG THE ELEMENTS AGAIN
     ----------------------------
      DO 16,I2=1,K
       I3=2*I2+K1
       ZB=B(I2)
       UNP1(I3)=ZB
 16   CONTINUE

     CALCULATE THE TIME STEP
     -----------------------
     ZUMAX=0.0
      DO 100 I=IIS,IIF
       ZUNP1=UNP1(I)
       AZ=ABS(ZUNP1)
       IF(AZ.GT.ZUMAX)ZUMAX=ZUNP1
 100  CONTINUE

     PROD=BETA*(ZUMAX**M)-2.0*JAMA/DXS
     APROD=ABS(PROD)
     IF(APROD.GT.0.00001) GO TO 18
     WRITE(5,17)APROD,ZUMAX
 17  FORMAT(10X,F12.5,5X,F10.5)
```

```
      STOP

18     DO 19,I=IIS,IIF
        UN(I)=UNP1(I)
        UNP1(I)=0.0
19     CONTINUE

      OUT PUT AT REGIRED TIME STEP
      ----------------------------
      IF(NSTEP.GT.NSTEPR) GOTO 20
        IF(((NSTEP/NSTEPS)*NSTEPR).EQ.(NSTEPR/NSTEPS)*NSTEP) GO TO 23
20       IF(((NSTEP/NSTEPR)*NSTEPF).NE.(NSTEPF/NSTEPR)*NSTEP) GOTO 28
      WRITE(5,21)NSTEP,DT,T
21    FORMAT(10X,'NSTEP=',I4,5X,'DT=',F8.5,5X,'T=',F8.5,10X)
      WRITE(5,22)
22    FORMAT(10X,'POSITION',6X,'NUM SOL',10X,'ANAL ASOL')

23    X=DXI
      ISM2=IIS-2
      IFP2=IIF+2
      DMBT=D-B1*T

        DO 25 ,I=ISM2,IFP2
         X=X+DX
         IF(M.EQ.4.OR.M.EQ.5.OR.M.EQ.6)THEN
          IF(I.GE.2.AND.I.LE.173)THEN
          ARG=A1*X+DMBT
          SSE=SECHS(ARG)
          ZUANA(I)=(A5*SSE)**(1.0/M)
          ZUANAL=ZUANA(I)
          ELSE
          ZUANAL=0.0
          END IF
         ELSE
         ARG=A1*X+DMBT
         SSE=SECHS(ARG)
         ZUANA(I)=(A5*SSE)**(1.0/M)
         ZUANAL=ZUANA(I)
         ENDIF
         ZUNUM=UN(I)
         WRITE(6,24)X,ZUNUM,ZUANAL
24      FORMAT(5X,F8.5,10X,F13.6,10X,F13.6)
25     CONTINUE

      CALCULATE L(INFINITY)
      MAX=0
       DO 26,I=IIS-2,IIF+2
        DD=UN(I)-ZUANA(I)
        AD=ABS(DD)
        IF(AD.GT.MAX) MAX=AD
26     CONTINUE
      WRITE(5,27)T,MAX
27     FORMAT(38X,F8.6,6X,F7.6)
      STABILITY CONDITION

28    SC=DTDDX*APROD
      NSTEP=NSTEP+1
      IF(SC.LE.1.0) GOTO 12
      WRITE(*,*)'SC=',SC,'DT=',DT
      WRITE(5,29)SC,DT,PROD,ZUMAX
29    FORMAT(10X,'SC=',F10.5,5X,'DT=',F10.5)
      STOP

30    M=M+DM
      IF(M.GT.2) GOTO 31
      GOTO 6
31    STOP

32    CLOSE(5)
      CLOSE(6)
      END
```